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Optimal with Respect to Accuracy Algorithms for Calculation of Multidimensional Weakly Singular Integrals and Applications to Calculation of Capacitances of Conductors of Arbitrary Shapes

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Key words: multidimensional weakly singular integrals, optimal quadrature rules, universal code, calculation of capacitance.

Abstract: Cubature formulas, asymptotically optimal with respect to accuracy, are derived for calculating multidimensional weakly singular integrals. They are used for developing a universal code for calculating capacitances of conductors of arbitrary shapes.

1. Introduction

Optimal with respect to accuracy methods for calculating singular integrals are being actively developed presently. They represent an important field of computational mathematics. Asymptotically optimal and optimal with respect to order (to accuracy and to complexity) algorithms for calculating singular integrals on closed and open contours, and multidimensional singular integrals have been constructed in [1-3] on Hölder and Sobolev classes of functions.

In constructing optimal with respect to accuracy methods for calculating one-dimensional, bisingular and multidimensional singular integrals, a general method, proposed in monograph [1], was used. This method can be applied not only to singular integrals but also to other integrals with moving singularities.

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This method allows one to construct several asymptotically optimal and optimal with respect to order and to accuracy algorithms for calculating hypersingular integrals [4], the Poisson and Cauchy type integrals [5], and multidimensional Cauchy type integrals.

Although multidimensional weakly singular integrals are used in many applications, optimal methods for calculating these integrals are not developed.

An exception is the book [1], where asymptotically optimal with respect to accuracy methods for calculating integrals of the form

$$\int_0^{2\pi}\int_0^{2\pi} f(\sigma_1,\sigma_2)\left|ctg\frac{\sigma_1-s_1}{2}\right|^{\gamma_1}\left|ctg\frac{\sigma_2-s_2}{2}\right|^{\gamma_2}d\sigma_1 d\sigma_2,$$

$0<\gamma_1,\ \gamma_2<1$, were constructed on Hölder and Sobolev classes.

Thus, the development of optimal methods for calculating multidimensional weakly singular integrals is an actual problem. Construction of efficient cubature formulas for calculating weakly singular integrals for calculating capacitances of conductors of arbitrary shapes by iterative methods proposed in [6] and [7] is very important in many applications, for example, in wave scattering by small bodies of arbitrary shapes and in antenna theory. A bibliography on methods for calculating capacitances and polarizability tensors is contained in [7].

In this paper the method proposed in [1] is generalized to multidimensional weakly singular integrals. As a result the analogs of the basic results for singular integrals, obtained earlier, are obtained for weakly singular integrals. Moreover, we study the applications of optimal with respect to order cubature formulas for calculating weakly singular integrals on Lyapunov surfaces. Our results are used for constructing an universal code for calculating capacitances and polarizability tensors of bodies of arbitrary shapes.

This paper consists of two parts.

In the first part of the paper optimal methods for calculating integrals of the types:

$$Kf\equiv\int_0^{2\pi}\int_0^{2\pi}\frac{f(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2}{\left(sin^2\left(\frac{\sigma_1-s_1}{2}\right)+sin^2\left(\frac{\sigma_2-s_2}{2}\right)\right)^{\lambda}},\quad 0\le s_1,s_2\le 2\pi; \tag{1.1}$$

and

$$Tf\equiv\int_{-1}^{1}\int_{-1}^{1}\frac{f(\tau_1,\tau_2)d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^{\lambda}},\quad -1\le t_1,t_2\le 1,\quad 0<\lambda<1, \tag{1.2}$$

are constructed on Hölder and Sobolev classes of functions.

Our results for integrals (1.1) can be generalized to the integrals with other periodic kernels and functions. The development of cubature formulas for integrals (1.1) is of considerable interest because the results are applicable to integrals with weakly singular kernels defined on closed Lyapunov surfaces.

It will be clear from our arguments, that the results can be generalized to $l-$dimensional integrals, $l = 3, 4, \cdots$.

The second part of this paper deals with the iterative methods for calculating capacitances of conductors of arbitrary shapes. A general numerical method for calculating these capacitances is developed, and the results of numerical tests are given.

2. Definitions of optimality.

Various definitions of optimality of numerical methods and a detailed bibliography can be found in [8,9,10]. Let us recall the definitions of algorithms, optimal with respect to accuracy, for calculating weakly singular integrals.

Consider the quadrature rule

$$Tf = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{l_1=0}^{\rho_1} \sum_{l_2=0}^{\rho_2} p_{k_1k_2l_1l_2}(t_1,t_2) f^{(l_1,l_2)}(x_{k_1}, y_{k_2}) +$$

$$+ R_{n_1n_2}(f; p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}; t_1, t_2), \tag{2.1}$$

where coefficients $p_{k_1k_2l_1l_2}(t_1,t_2)$ and nodes (x_{k_1}, y_{k_2}) are arbitrary. Here $f^{(l_1,l_2)}(s_1,s_2) = \partial^{l_1+l_2} f(s_1,s_2)/\partial s_1^{l_1} \partial s_2^{l_2}$.

The error of quadrature rule (2.1) is defined as

$$R_{n_1n_2}(f; p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}) = \sup_{(t_1,t_2)\in[-1,1]^2} |R_{n_1n_2}(f; p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}; t_1, t_2)|.$$

The error of quadrature rule (2.1) on the class Ψ is defined as

$$R_{n_1n_2}(\Psi; p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}) = \sup_{f\in\Psi} R_{n_1n_2}(f, p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}).$$

Define the functional

$$\zeta_{n_1n_2}(\Psi) = \inf_{p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}} R_{n_1n_2}(\Psi; p_{k_1k_2l_1l_2}; x_{k_1}, y_{k_2}).$$

The quadrature rule with the coefficients $p^*_{k_1k_2l_1l_2}$ and the nodes $(x^*_{k_1}, y^*_{k_2})$ is optimal, asymptotically optimal, optimal with respect to order on the class Ψ among all quadrature rules of type (2.1) provided that:

$$\frac{R_{n_1n_2}(\Psi; p^*_{k_1k_2l_1l_2}; x^*_{k_1}, y^*_{k_2})}{\zeta_{n_1n_2}(\Psi)} = 1, \sim 1, \asymp 1, \quad n_1, n_2 \to \infty.$$

The symbol $\alpha \asymp \beta$ means $A\alpha \le \beta \le B\alpha$, where $0 < A, B < \infty$.

Consider the quadrature rule

$$Tf = \sum_{k=1}^{n} p_k(t_1,t_2) f(M_k) +$$

$$+ R_n(f; p_k; M_k; t_1, t_2), \tag{2.2}$$

where coefficients $p_k(t_1,t_2)$ and nodes (M_k) are arbitrary.

The error of quadrature rule (2.2) is defined as

$$R_n(f;p_k;M_k) = \sup_{(t_1,t_2)\in[-1,1]^2} |R_n(f;p_k;M_k;t_1,t_2)|.$$

The error of quadrature rule (2.2) on the class Ψ is defined as

$$R_n(\Psi;p_k;M_k) = \sup_{f\in\Psi} R_n(f,p_k;M_k).$$

Define the functional

$$\zeta_n(\Psi) = \inf_{p_k;M_k} R_n(\Psi;p_k;M_k).$$

The quadrature rule with the coefficients p_k^* and the nodes (M_k^*) is optimal, asymptotically optimal, optimal with respect to order on the class Ψ among all quadrature rules of type (2.2) provided that:

$$\frac{R_n(\Psi;p_k^*;M_k^*)}{\zeta_n(\Psi)} = 1, \sim 1, \asymp 1, \quad n\to\infty.$$

By $R_{n_1n_2}(\Psi)$ the error of optimal cubature formulas on the class Ψ is defined. It is obvious that $R_{n_1n_2}(\Psi) = \zeta_{n_1n_2}(\Psi)$.

3. Classes of functions

In this section, we list several classes of functions which are used below. Some definitions are from [11,12].

A function f is defined on A=[a,b] or on A=K, where K is a unit circle, satisfies the Hölder condition with constant M and exponent α, or belongs to the class $H_\alpha(M), M>0,\ 0<\alpha\leq 1,$ if $|f(x')-f(x'')|\leq M|x'-x''|^\alpha$ for any $x',x''\in A$.

Class H_ω, where $\omega(h)$ is a modulus of continuity, consists of all functions $f\in C(A)$ with the property $|f(x_1)-f(x_2)|\leq M\omega(|x_1-x_2|), x_1,x_2\in A$.

Class $W^r(M)$ consists of functions $f\in C(A)$ which have continuous derivatives $f',f'',\dots,f^{(r-1)}$ on A, and a piecewise-continuous derivative $f^{(r)}$ on A satisfying $max_{x\in[a,b]}|f^{(r)}(x)|\leq M$.

Class $W_p^r(M),\ r=1,2\dots,1\leq p\leq\infty$, consists of functions $f(t)$, defined on a segment [a,b] or on $A=K$, that have continuous derivatives $f',f'',\dots,f^{(r-1)}$, and an integrable derivative $f^{(r)}$ such that

$$\left[\int_A |f^{(r)}(x)|^p dx\right]^{1/p} \leq M.$$

Class $W_\alpha^r(M),\ r=1,2\dots,0<\alpha\leq 1$, consists of functions $f(t)$, defined on a segment [a,b] or on $A=K$, which have continuous derivatives $f',f'',\dots,f^{(r)}$, such that

$$|f^{(r)}(x_1)-f^{(r)}(x_2)|\leq M|x_1-x_2|^\alpha.$$

A function $f(x_1,x_2,\dots,x_l), l=2,3,\dots,$ defined on $A=[a_1,b_2;a_2,b_2;$ $\dots;a_l,b_l]$ or on $A=K_1\times K_2\times\cdots\times K_l$, where $K_i,=1,2,\dots,l$, are unit circles, satisfying Hölder

conditions with constant M and exponent $\alpha_i, i = 1, 2, \ldots, l$, or belongs to the class $H_{\alpha_1,\ldots,\alpha_l}(M), M > 0, 0 < \alpha \leq 1, i = 1, 2, \ldots, l$, if

$$|f(x_1, x_2, \ldots, x_l) - f(y_1, y_2, \ldots, y_l)| \leq M(|x_1 - y_1|^{\alpha_1} + \cdots + |x_l - y_l|^{\alpha_l}).$$

Let ω, ω_i, where $i = 1, 2, \ldots, l, l = 1, 2, \ldots$, be moduli of continuity.

Class $H_{\omega_1,\ldots,\omega_l}(M)$, consists of all functions $f \in C(A), A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l$, with the property

$$|f(x_1, x_2, \ldots, x_l) - f(y_1, y_2, \ldots, y_l)| \leq M(\omega_1(|x_1 - y_1|) + \cdots + \omega_l(|x_l - y_l|)).$$

Let $H_j^\omega(A), j = 1, 2, 3, A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l, l = 2, 3, \ldots$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on A and such that

$$|f(x) - f(y)| \leq \omega(\rho_j(x, y)), j = 1, 2, 3,$$

where $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l), \rho_1(x, y) = max_{1 \leq i \leq l}(|x_i - y_i|), \rho_2(x, y) = \sum_{i=1}^l |x_i - y_i|, \rho_3(x, y) = [\sum_{i=1}^l |x_i - y_i|^2]^{1/2}$.

Let $H_j^\alpha(A), j = 1, 2, 3, A = [a_1, b_2; a_2, b_2; \ldots; a_l, b_l]$ or $A = K_1 \times K_2 \times \cdots \times K_l, l = 2, 3, \ldots$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on A and such that

$$|f(x) - f(y)| \leq (\rho_j(x, y))^\alpha, j = 1, 2, 3.$$

More general is the class $H_{\rho j}^\alpha(A), j = 1, 2, 3$. It consists of all functions $f(x)$ which can be represented as $f(x) = \rho(x)g(x)$, where $g(x) \in H_j^\alpha(A), j = 1, 2, 3$, and $\rho(x)$ is a nonnegative weight function.

Let $Z_j^\omega(A), j = 1, 2, 3$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on A and satisfying

$$|f(x) + f(y) - 2f((x + y)/2)| \leq \omega(\rho_j(x, y)/2), j = 1, 2, 3.$$

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Class $Z_{\rho j}^\alpha(A), j = 1, 2, 3$, consists of all functions $f(x)$ which can be represented as $f(x) = \rho(x)g(x)$, where $g(x) \in Z_j^\alpha(A), j = 1, 2, 3$, and $\rho(x)$ is a nonnegative weight function.

Let $W^{r_1,\ldots,r_l}(M), l = 1, 2, \ldots$, be the class of functions $f(x_1, x_2, \ldots, x_l)$ defined on a domain A, which have continuous partial derivatives
$\partial^{|v|} f(x_1, \ldots, x_l)/\partial x_1^{v_1} \ldots \partial x_l^{v_l}, 0 \leq |v| \leq r - 1, |v| = v_1 + \cdots + v_l, r_i \geq v_i \geq 0, i = 1, 2, \ldots, l, r = r_1 + \cdots + r_l$ and all piece-continuous derivatives of order r, satisfying $\|\partial^r f(x_1, \ldots, x_l)/\partial x_1^{r_1} \ldots \partial x_l^{r_l}\|_C \leq M$ and $\|\partial^{r_i} f(0, \ldots, 0, x_i, 0, \ldots, 0)/\partial x_i^{r_i}\|_C \leq M,\ i = 1, \ldots, l$.

Let $W_p^{r_1,\dots,r_l}(M), l=1,2,\dots, 1\le p\le\infty$ be the class of functions $f(x_1,x_2,\dots,x_l)$, defined on a domain $A=[a_1,b_1;\dots;a_l,b_l]$, with continuous partial derivatives $\partial^{|v|}f(x_1,\dots,x_l)/\partial x_1^{v_1}\dots\partial x_l^{v_l}, 0\le|v|\le r-1, |v|=v_1+\dots+v_l, r_i\ge v_i\ge 0, i=1,2,\dots,l, r=r_1+\dots+r_l$, and all derivatives of order r, satisfying

$$\|\partial^r f(x_1,\dots,x_l)/\partial x_1^{r_1}\partial x_2^{r_2}\dots\partial x_l^{r_l}\|_{L_p(A)}\le M,$$

$$\|\partial^{r_1+v_2+\dots+v_l}f(x_1,0,\dots,0)/\partial x_1^{r_1}\partial x_2^{v_2}\dots\partial x_l^{v_l}\|_{L_p([a_1,b_1])}\le M, |v_2|+|v_3|+\dots+|v_l|\le r-r_1-1;$$

$$\dots\dots\dots$$

$$\|\partial^{v_1+\dots+v_{l-1}+r_l}f(0,\dots,0,x_l)/\partial x_1^{v_1}\partial x_2^{v_2}\dots\partial x_{l-1}^{v_{l-1}}\partial x_l^{r_l}\|_{L_p([a_l,b_l])}\le M, |v_1|+|v_2|+\dots+|v_{l-1}|\le r-r_{l-1}-1.$$

Let $A=[a_1,b_2;a_2,b_2;\dots;a_l,b_l]$ or $A=K_1\times K_2\times\dots\times K_l$. Let $C^r(M)$ be the class of functions $f(x_1,x_2,\dots,x_l)$ which are defined in A and which have continuous partial derivatives of order r. Partial derivatives of order r satisfy the conditions

$$\|\frac{\partial^{|v|}f(x_1,\dots,x_l)}{\partial x_1^{v_1}\dots\partial x_l^{v_l}}\|_C\le M$$

for any $v=(v_1,\dots,v_l)$, where $v_i\ge 0, i=1,2,\dots,l$ are integer and $\sum_{i=1}^l v_i=r$.

By $\tilde\Psi$ we denote the set of periodic functions of the class Ψ.

It is known [13] that Lyapunov spheres are defined as regions bounded by a finite number of closed surfaces satisfying the three Lyapunov conditions:

1. At each point of the surface a tangent plane (and, therefore, a normal) exist.

2. If Θ is the angle between the normals at the points m_1 and m_2, and r is the distance between these points, then

$$\Theta<Ar^\lambda,\quad 0<\lambda\le 1,$$

where A and λ are positive numbers which do not depend on m_1 and m_2.

3. For all points of the surface, a number $d>0$ exists such that there is exactly one point at which a straight line, parallel to the normal at the surface point m, intersects the surface inside a sphere of radius d centered at m.

Let S be a Lyapunov sphere, and N be the exterior normal to this sphere. We introduce a local system of Cartesian coordinates (χ,η,ζ), whose origin is located at an arbitrary point m_0 of S, the ζ axis is directed along the normal N_0 at the point m_0, and the χ and η axes lie in the tangential plane. In a sufficiently small neighborhood of m_0, the equation of the surface S in the local coordinates (χ,η,ζ) has the form

$$\zeta=F(\chi,\eta).$$

Definition 4.1. **[13]** *The surface S belongs to the class $L_k(B,\alpha)$ if $F(\chi,\eta)\in W_\alpha^k(B)$, and the constants B and α do not depend on the choice of the point m_0.*

4. Auxiliary statements.

We need the following known facts from the theory of quadrature and cubature formulas. These facts can be found, for example, in [11],[12],[14], [15].

Lemma 4.1. *Let Ψ_1 be the class of functions $W_p^r(1)$, $1,2,\ldots,\quad 1\le p\le\infty$, $0\le t\le 1$, $f(t)\in\Psi_1$, and the quadrature rule*

$$\int\limits_0^1 f(t)dt=\sum_{k=1}^n p_k f(t_k)+R_n(f)$$

be exact on all the polynomials of order up to $p-1$, and has error $R_n(\Psi_1)$ on the class Ψ_1. Let Ψ_2 be the class of functions $W_p^r(1)$, $\quad r=1,2,\ldots,\quad 1\le p\le\infty,\quad a\le x\le b$, and $g(x)\in W_p^r(1)$. Then the quadrature rule

$$\int\limits_a^b g(x)dx=(b-a)\sum_{k=1}^n p_k g(a+(b-a)t_k)+R_n(g)$$

has error $R_n(\Psi_2)$ on the class of functions Ψ_2 and

$$R_n(\Psi_2)=(b-a)^{r+1-1/p}R_n(\Psi_1).$$

Theorem 4.1. **[11]** *Among quadrature formulas*

$$\int\limits_0^1 f(x)dx=\sum_{k=1}^m\sum_{l=0}^{\rho}p_{kl}f^{(l)}(x_k)+R(f)\equiv L(f)+R(f)$$

the best formula for the class $W_p^r(1)\quad(1\le p\le\infty)$ with $\rho=r-1$ and $r=1,2,\cdots$, or $\rho=r-2$ and $r=2,4,6,\cdots$, is the unique formula defined by the following nodes x_k^* and coefficients p_{kl}^*:

$$x_k^*=h(2(k-1)+[R_{rq}(1)]^{1/r}),\quad k=1,2,\ldots,m,$$

$$p_{kl}^*=(-1)^l p_{ml}^*=h^{l+1}\left\{\frac{(-1)^l}{(l+1)!}[R_{rq}(1)]^{(l+1)/r}+\frac{1}{r!}R_{rq}^{(r-1-1)}(1)\right\},$$

$$(l=0,1,\ldots,\rho),\quad p_{k,2v}^*=\frac{2h^{2v+1}}{r!}R_{rq}^{(r-2v-1)}(1),\quad \left(k=2,3,\ldots,m-1;\quad v=0,1,\ldots,\left[\frac{r-1}{2}\right]\right),$$

$$p_{k,2v+1}^*=0\left(k=2,3,\ldots,m-1;\quad v=0,1,\ldots,\left[\frac{r-2}{2}\right]\right),\quad h=2^{-1}(m-1+[R_{rq}(1)]^{1/r})^{-1},$$

and $R_{rq}(t)$ is the Chebyshev polynomial $t^r+\sum\limits_{i=0}^{r-1}\beta_i t^i$, deviating least from zero in the norm $L_q(-1,1)$, where $p^{-1}+q^{-1}=1$. Here

$$\zeta_n[W_p^r(1)]=R_n[W_p^r(1)]=\frac{R_{rq}(1)}{2^r r!\sqrt[q]{rq+1}(m-1+[R_{rq}(1)]^{1/r})^r}.$$

Let a function $f(x,y)$ be given on a rectangle $D=[a,b;c,d]$. Consider the cubature formula

$$\iint\limits_D f(x,y)dxdy=\sum_{k=1}^m\sum_{i=1}^n p_{ki}f(x_k,y_i)+R_{mn}(f), \tag{4.1}$$

defined by a vector (X, Y, P) of a nodes $a \le x_1 < x_2 < \cdots < x_m \le b$, $c \le y_1 < y_2 < \cdots < y_n \le d$, and coefficients p_{ki}.

Theorem 4.2 [11]. *Among all quadrature formulas of the form of (4.1) the formula*

$$\iint_D f(x,y)dxdy = 4hq \sum_{k=1}^{m} \sum_{i=1}^{n} f(a + (2k-1)h, c + (2i-1)q) + R_{mn}(f),$$

where $h = \frac{b-a}{2m}$, $q = \frac{d-c}{2n}$, *is optimal on the classes* $H_{\omega_1,\omega_2}(D)$ *and* $H_3^{\omega}(D)$. *In addition*

$$R_{mn}[H_{\omega_1,\omega_2}(D)] = 4mn[q \int_0^h \omega_1(t)dt + h \int_0^q \omega_2(t)dt];$$

$$R_{mn}[H_3^{\omega}(D)] = 4mn \int_0^q \int_0^h \omega(\sqrt{t^2+\tau^2})dtd\tau.$$

Consider the cubature formulas of the form:

$$\iint_D p(x,y)f(x,y)dxdy = \sum_{k=1}^{N} p_k f(M_k) + R(f), \tag{4.2}$$

where $p(x, y)$ is a nonnegative and bounded on D function, p_k, $M_k(M_k \in D)$ are coefficients and nodes.

Theorem 4.3 [11]. *Let* $p(x, y)$ *be a nonnegative bounded weight function. If* $R_N[H^{\alpha}_{p,j}(D)]$ *and* $R_N[Z^{\alpha}_{p,j}(D)]$, *where* $j = 1, 2, 3$, *and* $0 < \alpha \le 1$, *are the errors of optimal formulas as (4.2) on the classes* $H^{\alpha}_{p,j}(D)$ *and* $Z^{\alpha}_{p,j}(D)$, *respectively, then*

$$\lim_{N\to\infty} N^{\alpha/2} R_N[H^{\alpha}_{p,j}(D)] = \lim_{N\to\infty} N^{\alpha/2} 2R_N[Z^{\alpha}_{p,j}(D)] =$$

$$= D_j \left[\int\int_D (p(x,y))^{2/(2+\alpha)} dxdy \right]^{(2+\alpha)/\alpha}, \quad j = 1, 2, 3,$$

where $D_1 = \frac{12}{2+\alpha} \left(\frac{1}{2\sqrt{3}}\right)^{(2+\alpha)/\alpha} \int_0^{\pi/6} \frac{d\varphi}{cos^{2+\alpha}\varphi}$, $D_2 = 2^{1-\alpha}/(2+\alpha)$, *and* $D_3 = 2^{1-\alpha/2}/(2+\alpha)$.

If $j = 2$, *then the conclusion holds for* $n-$*dimensional cubature formulas.*

Remark. *Theorem 4.2 is generalized to the case of unbounded weights* $p(x, y)$ *in [2].*

In this paper we will use the following result (see [16]):

Lemma 4.4 . *Let* H *be a linear metric space,* F *be a bounded, closed, convex, centrally symmetric set with center of symmetry* θ *at the origin, and* $L(f), l_1(f), \ldots, l_N(f)$, *be some linear functionals. Let* $S(l_1(f), \ldots, l_N(f))$ *be some method for calculating the functional* $L(f)$ *using functionals* $(l_1(f), \ldots, l_N(f))$, *and* $\mathcal{S}$ *be the set of all such methods. Then the numbers* $D_1, \ldots, D_N$ *exist such that*

$$\sup_{f\in F} |L(f) - \sum_{k=1}^{N} D_k l_k(f)| = \inf_{\mathcal{S}} \sup_{f\in F} |L(f) - S(l_1(f), \ldots, l_N(f))|. \tag{4.3}$$

This means that among the best methods for calculating functional $L(f)$:

$$L(f) \approx S(l_1(f), \ldots, l_N(f)), \tag{4.4}$$

there is a linear method.

Proof. Let us associate with each $f \in F$ a point $(L(f), l_1(f), \ldots, l_N(f))$. Let Y be a set of all such points $(y_0, \ldots, y_N)$ for $f \in F$.

From our assumptions, it follows that Y is a closed centrally symmetric set with the center of symmetry at the origin.

Let $(y_0, 0, \ldots, 0)$ be an extremal point of the set Y, and

$$D_0 = \sup_{(z,0,\ldots,0)\in Y} z = y_0.$$

Because F is bounded, one has $D_0 < \infty$, and because F is convex and centrally symmetric with respect to the origin, one has $D_0 > 0$.

Draw the support plane for the set Y through the point $(D_0, 0, \ldots, 0)$:

$$(y_0 - D_0) + \sum_{j=1}^{N} C_j y_j = 0.$$

Since Y is centrally symmetric with respect to the origin, the plane

$$(y_0 + D_0) + \sum_{j=1}^{N} C_j y_j = 0$$

is also a support plane for Y, and Y lies between these two planes.

Hence, we have for the points of Y the inequality:

$$|y_0 - \sum_{j=1}^{N} D_j y_i| \leq D_0, \quad D_j = -C_j.$$

The definition of y_i implies

$$\sup_{f\in F} |L(f) - \sum_{j=1}^{N} D_j l_j(f)| \leq D_0. \tag{4.5}$$

Let f_0 be an element F corresponding the point $(D_0, 0, \ldots, 0)$. Then $S(l_1(\pm f_0), \ldots, l_N(\pm f_0)) = S(0, \ldots, 0)$. The right-hand side of (4.3) is not less than

$$\inf_{\mathcal{S}} \max |L(f_0) - S(0, ..., 0)| = \inf_{a} \max\{|D_0 - a|, |D_0 + a|\} = D_0.$$

This and (4.5) imply that the right-hand side in (4.3) is not less that the left-hand one. But the right-hand side of (4.3) can not be more than the left-hand side of (4.3) because a set of methods $\mathcal{S}$ contains linear methods. Lemma 4.4 is proved. ■

Corollary. *Among all functions for which the optimal method for calculating $L(t)$ has the greatest error for a given set of functionals, there exists a function satisfying the conditions $l_1(f) = \cdots = l_N(f) = 0$.* It follows from the proof that such a function is the function f_0.

5. Optimal methods for calculating integrals of the form (1.1).

5.1. Lower bounds for the functionals ζ_{nm} and ζ_N.

In this Section we derive lower bounds for the functionals ζ_{nm} and ζ_N, defined in Section 2 , for calculating integrals (1.1) by the cubature formulas

$$Kf = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{l_1=0}^{\rho_1} \sum_{l_2=0}^{\rho_2} p_{k_1k_2l_1l_2}(s_1, s_2) f^{(l_1,l_2)}(x_{k_1}, x_{k_2}) +$$

$$+ R_{n_1n_2}(f; p_{k_1k_2l_1l_2}; x_{k_1}, x_{k_2}; s_1, s_2), \tag{5.1}$$

and

$$Kf = \sum_{k=1}^{N} p_k(s_1, s_2) f(M_k) + R_N(f; p_k; M_k; s_1, s_2) \tag{5.2}$$

on Hölder and Sobolev classes.

Theorem 5.1. *Let $\Psi = H_{\omega_1,\omega_2}(D)$ or $\Psi = H_3^{\omega}(D)$, and calculate integral (1.1) by formula (5.1) with $\rho_1 = \rho_2 = 0$. Then the inequality*

$$\zeta_{n_1n_2}[\Psi] \geq \frac{\gamma}{\pi^2} n_1 n_2 [q \int\limits_0^h \omega_1(t)dt + h \int\limits_0^q \omega_2(t)dt],$$

where $q = \frac{\pi}{n_2}$, $h = \frac{\pi}{n_1}$, and

$$\gamma := \int\limits_0^{2\pi} \int\limits_0^{2\pi} \frac{ds_1 ds_2}{(sin^2(s_1/2) + sin^2(s_2/2))^{\lambda}} \tag{5.1$'$}$$

is valid.

Corollary. *Let $\Psi = H_{\alpha\alpha}(D)$ or $\Psi = H_3^{\alpha}(D)$, and calculate integral (1.1) by formula (5.1) with $n_1 = n_2 = n$ and $\rho_1 = \rho_2 = 0$. Then the inequality*

$$\zeta_{nn}[\Psi] \geq \frac{2\gamma\pi^{\alpha}}{(1+\alpha)n^{\alpha}}$$

is valid.

Proof of Theorem 5.1. Denote by $\psi(s_1, s_2)$ a nonnegative function belonging to the class $H_{\omega_1\omega_2}(1)$ and vanishing at the nodes (x_{k_1}, x_{k_2}), $1 \leq k_1 \leq n_1$, $1 \leq k_2 \leq n_2$.

One has:

$$R_{n_1n_2}(\psi; p_{k_1k_2}; x_{k_1}, x_{k_2}) \geq$$

$$\geq \frac{1}{4\pi^2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} \left(\int\limits_0^{2\pi} \int\limits_0^{2\pi} \frac{\psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2}{[sin^2((\sigma_1 - s_1)/2) + sin^2((\sigma_2 - s_2)/2))]^{\lambda}} \right) ds_1 ds_2 =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\psi(\sigma_1,\sigma_2)\left(\int_0^{2\pi}\int_0^{2\pi}\frac{ds_1ds_2}{[sin^2((\sigma_1-s_1)/2)+sin^2((\sigma_2-s_2)/2))]^\lambda}\right)d\sigma_1d\sigma_2 =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{ds_1ds_2}{[sin^2(s_1/2)+sin^2(s_2/2)]^\lambda}\int_0^{2\pi}\int_0^{2\pi}\psi(s_1,s_2)ds_1ds_2. \tag{5.3}$$

From Lemma 4.4 and Theorem 4.2 one concludes that the following inequality

$$R_{n_1n_2}(\psi;p_{k_1k_2};x_{k_1},x_{k_2}) \geq \frac{\gamma}{\pi^2}n_1n_2\left[q\int_0^h\omega_1(t)dt + h\int_0^q\omega_2(t)dt\right], h=\frac{\pi}{n_1}, \quad q=\frac{\pi}{n_2}$$

holds for arbitrary weights $p_{k_1k_2}$ and nodes (x_{k_1},x_{k_2}) and

$$\zeta_{nn}(\Psi) \geq \frac{\gamma}{\pi^2}n_1n_2[q\int_0^h\omega_1(t)dt + h\int_0^q\omega_2(t)dt].$$

Theorem 5.1 is proved. ■

Theorem 5.2. *Let $\Psi = H_i^\alpha$ or $\Psi = Z_i^\alpha$, $i=1,2,3$, and calculate the integral Kf by cubature formula (5.2). Then*

$$\zeta_N[H_i^\alpha] = 2\zeta_N[Z_i^\alpha] = (1+o(1))\gamma(4\pi^2)^{2/\alpha}D_iN^{-\alpha/2},$$

where $D_1 = \frac{12}{2+\alpha}\left(\frac{1}{2\sqrt{3}}\right)^{(\alpha+2)/2}\int_0^{\pi/6}\frac{d\varphi}{cos^{2+\alpha}\varphi}$, $D_2 = \frac{2}{2^\alpha(2+\alpha)}$, *and* $D_3 = \frac{2^{1-\alpha/2}}{2+\alpha}$.

Proof. The proof of Theorem 5.2 is similar to the proof of Theorem 5.1, with some difference is in the estimation of the integral $\int_0^{2\pi}\int_0^{2\pi}\psi(s_1,s_2)ds_1ds_2$, where the function $\psi(s_1,s_2)$ belongs to the class H_i^α (or Z_i^α), is nonnegative in the domain $D=[0,2\pi]^2$, and vanishes at N nodes M_k, $k=1,2,\dots,N$.

Using Lemma 4.4 and Theorem 4.3, one checks that the inequalities

$$\inf_{M_k}\sup_{\psi\in H_i^\alpha,\psi(M_k)=0}\int_0^{2\pi}\int_0^{2\pi}\psi(s_1,s_2)ds_1ds_2 = (1+o(1))D_i(4\pi^2)^{(2+\alpha)/\alpha}N^{-\alpha/2},$$

$$\inf_{M_k}\sup_{\psi\in Z_i^\alpha,\psi(M_k)=0}\int_0^{2\pi}\int_0^{2\pi}\psi(s_1,s_2)ds_1ds_2 = (1+o(1))\frac{1}{2}D_i(4\pi^2)^{(2+\alpha)/\alpha}N^{-\alpha/2}$$

hold for arbitrary $M_k\in D$, $k=1,2,\dots,N$.

Substituting these values into inequality (5.3), we complete the proof of Theorem 5.2. ■

Theorem 5.3. *Let $\Psi = \tilde{C}_2^r(1)$, and calculate the integral Kf by formula (5.1) with $\rho_1=\rho_2=0$, and $n_1=n_2=n$. Then*

$$\zeta_{nn}[\Psi] \geq (1+o(1))\frac{2\gamma K_r}{n^r},$$

where K_r is the Favard constant.

Proof. Let

$$\psi(s_1,s_2) = \psi_1(s_1)+\psi_2(s_2),$$

where $0 \le \psi_1(s) \in W^r(1)$ vanishes at the nodes x_k, $k = 1, 2, \dots, n$, and $0 \le \psi_2(s) \in W^r(1)$ vanishes at the nodes y_k, $k = 1, 2, \dots, n$.

According to [11], for arbitrary nodes x_k, $k = 1, 2, \dots, n$ one has:

$$\int_0^{2\pi} \psi_i(s)ds \ge \frac{2\pi K_r}{n^r}, \ i = 1, 2.$$

Thus, the inequality

$$\int_0^{2\pi}\int_0^{2\pi} \psi(s_1, s_2)ds_1 ds_2 \ge \frac{8\pi^2 K_r}{n^r}$$

holds for arbitrary nodes $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$.

The conclusion of Theorem 5.3 follows from this inequality and from (5.3). ■

Theorem 5.4. *Let* $\Psi = W_p^{r,r}(1)$, $r = 1, 2, \dots$, $1 \le p \le \infty$, *and calculate the integral* Kf *by formula (5.1) with* $\rho_1 = \rho_2 = r - 1$ *and* $n_1 = n_2 = n$. *Then*

$$\zeta_{nn}[\Psi] \ge (1 + o(1)) \frac{2^{1/q}\pi^{r-1/p} R_{rq}(1)}{r!(rq+1)^{1/q}(n - 1 + [R_{rq}(1)]^{1/r})^r}\gamma,$$

where $R_{rq}(t)$ *is a polynomial of degree* r, *least deviating from zero in* $L_q([-1, 1])$.

Proof. Let $L = [\frac{n}{\log n}]$. Take an additional set of nodes (ξ_k, ξ_l), $\xi_k = \frac{2\pi k}{L}$, $k, l = 0, 1, \dots, L - 1$. By (v_i, w_j), $i, j = 0, 1, \dots, N - 1$, $N = n + L$, denote the union of the sets (x_k, y_l) and (ξ_i, ξ_j). Let $\psi(s_1, s_2) = \psi_1(s_1) + \psi_2(s_2)$, where $\psi_1(s) \in W_p^r(1)$ vanishes with its derivatives up to the order $r - 1$ at the nodes v_i, $i = 0, 1, \dots, N - 1$, and $\psi_2(s) \in W_p^{(r)}(1)$ vanishes with its derivatives up to order $r - 1$ at the nodes w_j, $j = 0, 1, \dots, N - 1$. Assume $\int\limits_{v_i}^{v_{i+1}} \psi_1(s)ds > 0$, $i = 0, 1, \dots, N - 1$, and $\int\limits_{w_j}^{w_{j+1}} \psi_2(s)ds > 0$, $j = 0, 1, \dots, N - 1$, where $v_N = 2\pi$ and $w_N = 2\pi$.

Let

$$\mathbf{h}(\mathbf{s_1}, \mathbf{s_2}, \boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}) := \begin{cases} 0, \quad if \quad (\sigma_1, \sigma_2) = (s_1, s_2), \\ \frac{1}{(sin^2((\sigma_1 - s_1)/2) + sin^2((\sigma_2 - s_2)/2))^\lambda}, \quad otherwise, \end{cases}$$

$$\psi^+(\mathbf{s_1}, \mathbf{s_2}) = \begin{cases} \psi(s_1, s_2), \quad if \quad \psi(s_1, s_2) \ge 0, \\ 0, \quad if \quad \psi(s_1, s_2) < 0, \end{cases}$$

$$\psi^-(\mathbf{s_1}, \mathbf{s_2}) = \begin{cases} 0, \quad if \quad \psi(s_1, s_2) \ge 0, \\ -\psi(s_1, s_2), \quad if \quad \psi(s_1, s_2) < 0. \end{cases}$$

For each value (ξ_i, ξ_j), $i, j = 0, 1, \dots, N - 1$, we have (with $N = N_1 = N_2 = L$):

$$\int_0^{2\pi}\int_0^{2\pi} h(\xi_i, \xi_j, \sigma_1, \sigma_2)\psi(\sigma_1, \sigma_2)d\sigma_1 d\sigma_2 =$$

$$= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \int_{\xi_k}^{\xi_{k+1}}\int_{\xi_l}^{\xi_{l+1}} h(\xi_i, \xi_j, \sigma_1, \sigma_2)\psi(\sigma_1, \sigma_2)d\sigma_1 d\sigma_2 =$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_l}^{\xi_{l+1}} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$- \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_l}^{\xi_{l+1}} h(\xi_i, \xi_j, \sigma_1, \sigma_2) \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \geq$$

$$\geq \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_l}^{\xi_{l+1}} \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l-1}) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_{l-1}}^{\xi_l} \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_l}^{\xi_{l+1}} \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l-1}) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_{l-1}}^{\xi_l} \psi^+(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$- \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_k, \xi_l) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_l}^{\xi_{l+1}} \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$- \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_k, \xi_l) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_{l-1}}^{\xi_l} \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$- \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_k, \xi_l) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_l}^{\xi_{l+1}} \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$- \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_k, \xi_l) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_{l-1}}^{\xi_l} \psi^-(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 =$$

$$= \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i+1}^{i+[(N_1-1)/2]} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l-1}) \int\limits_{\xi_k}^{\xi_{k+1}} \int\limits_{\xi_{l-1}}^{\xi_l} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j+1}^{j+[(N_2-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(N_1-1)/2]}^{i-1} \sum_{l=j-[(N_2-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l-1}) \int\limits_{\xi_{k-1}}^{\xi_k} \int\limits_{\xi_{l-1}}^{\xi_l} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 -$$

$$-\sum_{k=i+1}^{i+[(N_1-1)/2]}\sum_{l=j+1}^{j+[(N_2-1)/2]}(h(\xi_i,\xi_j,\xi_k,\xi_l)-h(\xi_i,\xi_j,\xi_{k+1},\xi_{l+1}))\times$$

$$\times\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}\psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2-$$

$$-\sum_{k=i+1}^{i+[(N_1-1)/2]}\sum_{l=j-[(N_2-1)/2]}^{j-1}(h(\xi_i,\xi_j,\xi_k,\xi_l)-h(\xi_i,\xi_j,\xi_{k+1},\xi_{l-1}))\times$$

$$\times\int\limits_{\xi_{k+1}}^{\xi_k}\int\limits_{\xi_{l-1}}^{\xi_l}\psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2-$$

$$-\sum_{k=i-[(N_1-1)/2]}^{i-1}\sum_{l=j+1}^{j+[(N_2-1)/2]}(h(\xi_i,\xi_j,\xi_k,\xi_l)-h(\xi_i,\xi_j,\xi_{k-1},\xi_{l+1}))\times$$

$$\times\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_l}^{\xi_{l+1}}\psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2-$$

$$-\sum_{k=i-[(N_1-1)/2]}^{i-1}\sum_{l=j-[(N_2-1)/2]}^{l=j-1}(h(\xi_i,\xi_j,\xi_k,\xi_l)-h(\xi_i,\xi_j,\xi_{k-1},\xi_{l-1}))\times$$

$$\times\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_{l-1}}^{\xi_l}\psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2=$$

$$=J_1+J_2+J_3+J_4+I_1+I_2+I_3+I_4.$$

Let us estimate the integral

$$\left|\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}\psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right|\le\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}|\psi^-(\sigma_1,\sigma_2)|d\sigma_1 d\sigma_2\le$$

$$\le\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}|\psi(\sigma_1,\sigma_2)|d\sigma_1 d\sigma_2\le(\xi_{l+1}-\xi_l)\int\limits_{\xi_k}^{\xi_{k+1}}|\psi_1(\sigma)|d\sigma+(\xi_{k+1}-\xi_k)\int\limits_{\xi_l}^{\xi_{l+1}}|\psi_2(\sigma)|d\sigma\le$$

$$\le 2\left(\frac{2\pi}{L}\right)^{r+2}\frac{1}{r!},$$

where we have used the fact that the functions $\psi_1(s)$ and $\psi_2(s)$ on the segments $[\xi_k,\xi_{k+1}]$ and $[\xi_l,\xi_{l+1}]$ vanish with derivatives up to order $r-1$.

Now let us estimate the sum:

$$\sum_{k=i+1}^{i+[(L-1)/2]}\sum_{l=j+1}^{j+[(L-1)/2]}|h(\xi_i,\xi_j,\xi_{k+1},\xi_{l+1})-h(\xi_i,\xi_j,\xi_k,\xi_l)|=$$

$$= \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{j+[(L-1)/2]} \left| \frac{1}{\left(sin^2\frac{\pi(k+1-i)}{L} + sin^2\frac{\pi(l+1-j)}{L}\right)^{\lambda}} - \right.$$

$$\left. - \frac{1}{\left(sin^2\frac{2\pi(k-i)}{L} + sin^2\frac{2\pi(l-j)}{L}\right)^{\lambda}} \right| \leq$$

$$\leq \frac{c}{L} \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{j+[(L-1)/2]} \frac{1}{\left(sin^2\frac{\pi(k-i)}{L} + sin^2\frac{\pi(l-j)}{L}\right)^{1+\lambda}} \left| \frac{k-i}{L} + \frac{l-j}{L} \right| \leq$$

$$\leq \frac{c}{L} \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{j+[(L-1)/2]} \frac{L^{2+2\lambda}}{((k-i)^2 + (l-j)^2)^{1+\lambda}} \frac{(k-i)+(l-j)}{L} \leq$$

$$\leq c\,(L)^{2\lambda} \left(\sum_{l=1}^{[(L-1)/2]} \frac{1}{l^{2\lambda}} + \sum_{k=1}^{[(L-1)/2]} \frac{1}{k^{2\lambda}} \right) \leq$$

$$\leq c\,(L)^{2\lambda} \begin{cases} L^{1-2\lambda} & if \quad \lambda < \frac{1}{2}, \\ \log L & if \quad \lambda = \frac{1}{2}, \\ 1 & if \quad \lambda > \frac{1}{2}. \end{cases}$$

By $c > 0$ various estimation constants are denoted. Thus

$$I_1 = o\left(\frac{1}{n^r}\right).$$

The expressions I_2, I_3, and I_4 are esimated similarly.

From the definition of the function $\psi(s_1, s_2)$ it follows that the error of cubature formula (5.1) for $s_1 = \xi_i$, $s_2 = \xi_j$ can be estimated as follows:

$$R(\psi,, \xi_i, \xi_j) = \int_0^{2\pi}\int_0^{2\pi} \psi(\sigma_1, \sigma_2) h(\xi_i, \xi_j, \sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \geq o\left(\frac{1}{n^r}\right) +$$

$$+ \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j+1}^{j+[(L-1)/2]} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l+1}) \int_{\xi_k}^{\xi_{k+1}} \int_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i+1}^{i+[(L-1)/2]} \sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k+1}, \xi_{l-1}) \int_{\xi_k}^{\xi_{k+1}} \int_{\xi_{l-1}}^{\xi_l} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(L-1)/2]}^{i-1} \sum_{l=j+1}^{j+[(L-1)/2]} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l+1}) \int_{\xi_{k-1}}^{\xi_k} \int_{\xi_l}^{\xi_{l+1}} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 +$$

$$+ \sum_{k=i-[(L-1)/2]}^{i-1} \sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i, \xi_j, \xi_{k-1}, \xi_{l-1}) \int_{\xi_{k-1}}^{\xi_k} \int_{\xi_{l-1}}^{\xi_l} \psi(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2.$$

Averaging the above inequality over i and j, one gets:

$$R_{nn}[\Psi] \geq \sup_{\psi\in\Psi} \max_{i,j} R_{nn}(\psi,,\xi_i,\xi_j) \geq$$

$$\geq \frac{1}{L^2}\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\left[\sum_{k=i+1}^{i+[(L-1)/2]}\sum_{l=j+1}^{j+[(L-1)/2]} h(\xi_i,\xi_j,\xi_{k+1},\xi_{l+1})\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2+\right.$$

$$+\sum_{k=i+1}^{i+[(L-1)/2]}\sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i,\xi_j,\xi_{k+1},\xi_{l-1})\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_{l-1}}^{\xi_l}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2+$$

$$+\sum_{k=i-[(L-1)/2]}^{i-1}\sum_{l=j+1}^{j+[(L-1)/2]} h(\xi_i,\xi_j,\xi_{k-1},\xi_{l+1})\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_l}^{\xi_{l+1}}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2+$$

$$\left.+\sum_{k=i-[(L-1)/2]}^{i-1}\sum_{l=j-[(L-1)/2]}^{j-1} h(\xi_i,\xi_j,\xi_{k-1},\xi_{l-1})\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_{l-1}}^{\xi_l}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\right]+o\left(\frac{1}{n^r}\right)\geq$$

$$\geq o\left(\frac{1}{n^r}\right)+$$

$$+\frac{1}{L^2}\left[\sum_{k=i+1}^{i+[(L-1)/2]}\sum_{l=j+1}^{j+[(L-1)/2]}\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_l}^{\xi_{l+1}}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}h(\xi_i,\xi_j,\xi_{k+1},\xi_{l+1})+\right.$$

$$+\sum_{k=i+1}^{i+[(L-1)/2]}\sum_{j-[(L-1)/2]}^{l=j-1}\int\limits_{\xi_k}^{\xi_{k+1}}\int\limits_{\xi_{l-1}}^{\xi_l}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}h(\xi_i,\xi_j,\xi_{k+1},\xi_{l-1})+$$

$$+\sum_{k=i-[(L-1)/2]}^{i-1}\sum_{l=j+1}^{j+[(L-1)/2]}\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_l}^{\xi_{l+1}}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}h(\xi_i,\xi_j,\xi_{k-1},\xi_{l+1})+$$

$$\left.+\sum_{k=i-[(L-1)/2]}^{i-1}\sum_{l=j-[(L-1)/2]}^{j-1}\int\limits_{\xi_{k-1}}^{\xi_k}\int\limits_{\xi_{l-1}}^{\xi_l}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}h(\xi_i,\xi_j,\xi_{k-1},\xi_{l-1})\right]=$$

$$=o(\frac{1}{n^r})+\frac{1}{4\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2\times$$

$$\times\left(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\frac{d\sigma_1 d\sigma_2}{(sin^2(\sigma_1/2)+sin^2(\sigma_2/2))^{\lambda}}+O\left(\left(\frac{logn}{n}\right)^{2-2\lambda}\right)\right), \tag{5.4}$$

where the following relation was used:

$$\frac{4\pi^2}{L^2}\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}h(\xi_i,\xi_j,\xi_{k-1},\xi_{l-1})=O\left(\frac{\log n}{n}\right)+\int\limits_0^{2\pi}\int\limits_0^{2\pi}\frac{d\sigma_1 d\sigma_2}{[sin^2(\sigma_1/2)+sin^2(\sigma_2/2)]^{\lambda}}.$$

Without loss of generality one may assume $k=1, l=1$ in the previous equation. Let us estimate

$$U_0 = \left| \frac{4\pi^2}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} h(\xi_i, \xi_j,\ 0, 0) - \int_0^{2\pi}\int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{(sin^2(\sigma_1/2) + sin^2(\sigma_2/2))^\lambda} \right| \leq$$

$$\leq \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1}{}' \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \left[\frac{1}{(sin^2((\xi_i)/2) + sin^2((\xi_j)/2))^\lambda} - \right.\right.$$

$$\left.\left. - \frac{1}{(sin^2(\sigma_1/2) + sin^2(\sigma_2/2))^\lambda} \right] d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \int_0^{\xi_1}\int_0^{\xi_1} \frac{1}{(sin^2(\sigma_1/2) + sin^2(\sigma_2/2))^\lambda} d\sigma_1 d\sigma_2 \right| = u_1 + u_2,$$

where $\sum\sum{}'$ means summation over $(i,j) \neq (0,0)$.

Let us estimate u_1 and u_2. One has:

$$u_1 \leq \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1}{}' \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \left[\frac{1}{(sin^2((\sigma_1)/2) + sin^2((\sigma_2)/2))^\lambda} - \right.\right.$$

$$\left.\left. - \frac{1}{(sin^2(\xi_i/2) + sin^2(\sigma_2/2))^\lambda} \right] d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \sum_{i=0}^{L-1} \sum_{j=0}^{L-1}{}' \int_{\xi_i}^{\xi_{i+1}} \int_{\xi_j}^{\xi_{j+1}} \left[\frac{1}{(sin^2((\xi_i)/2) + sin^2((\sigma_2)/2))^\lambda} - \right.\right.$$

$$\left.\left. - \frac{1}{(sin^2(\xi_i/2) + sin^2(\xi_j/2))^\lambda} \right] d\sigma_1 d\sigma_2 \right| =$$

$$= u_{11} + u_{12}.$$

The expressions u_{11} and u_{12} can be estimated similarly. Let us estimate u_{11}:

$$u_{11} \leq \frac{c}{L^4} \sum_{i=0}^{L} \sum_{j=0}^{L}{}' \frac{1}{(sin^2((\xi_i)/2) + sin^2((\xi_j)/2))^{1+\lambda}} \leq$$

$$\leq \frac{c}{L^{2-2\lambda}} \sum_{i=0}^{L} \sum_{j=0}^{L}{}' \frac{1}{(i^2 + j^2)^{1+\lambda}} \leq c \frac{1}{L^{2-2\lambda}},$$

where $c > 0$ stands for various estimation constants. Hence

$$u_1 \leq \frac{c}{L^{2-2\lambda}}.$$

Let us estimate u_2 :

$$u_2 = \left| \int_0^{\xi_1}\int_0^{\xi_1} \frac{1}{(sin^2(\sigma_1/2) + sin^2(\sigma_2/2))^\lambda} d\sigma_d \sigma_2 \right| \leq$$

$$\leq c\int_0^{\xi_1}\int_0^{\xi_1}\frac{1}{\left(\sigma_1^2+\sigma_2^2\right)^{\lambda}}d\sigma_1 d\sigma_2.$$

Using polar coordinates, one gets:

$$u_2\leq c\int_0^{1/L}\int_0^{2\pi}\frac{1}{\rho^{2\lambda-1}}d\rho d\phi\leq\frac{c}{L^{2-2\lambda}}.$$

Thus:

$$U_0\leq\frac{c}{L^{2-2\lambda}}.$$

From Lemmas 4.4, 4.1, and Theorem 4.1 it follows that

$$\int_0^{2\pi}\psi_1(\sigma_1)d\sigma_1\geq\frac{(1+o(1))(2\pi)^{r+1/q}R_{rq}(1)}{2^r r!(rq+1)^{1/q}(n-1+[R_{rq}(1)]^{1/r})^r},\tag{5.5}$$

where $R_{rq}(t)$ is a polynomial of degree r, least deviating from zero in $L_q([-1,1])$.

Theorem 5.4 follows from inequalities (5.4) and (5.5).

■

5.2. Optimal cubature formulas for calculating integrals (1.1).

Hölder class of functions.

Let $x_k := 2k\pi/n,\ k=0,1,\dots,n,\ \Delta_{kl}=[x_k,x_{k+1},x_l,x_{l+1}],\ k,l=0,1,\dots,n-1,\ x'_k=(x_{k+1}+x_k)/2,\ k=0,1,\dots,n-1$, and $(s_1,s_2)\in\Delta_{ij},\ i,j=0,1,\dots,n-1$.

Calculate the integral Kf by the formula:

$$Kf=\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}f(x'_k,x'_l)\int\int_{\Delta_{kl}}\frac{d\sigma_1 d\sigma_2}{\left(sin^2\left(\frac{\sigma-x'_i}{2}\right)+sin^2\left(\frac{\sigma-x'_j}{2}\right)\right)^{\lambda}}+R_{nn}.\tag{5.6}$$

Theorem 5.5. *Let $\Psi=H_{\alpha\alpha}(D), 0<\alpha<1$. Among all cubature formulas (5.1) with $\rho_1=\rho_2=0$, formula (5.6), which has the error*

$$R_{nn}[\Psi]=\frac{(2+o(1))\gamma}{1+\alpha}\left(\frac{\pi}{n}\right)^{\alpha},$$

is asymptotically optimal. Here γ is defined in (5.1').

Proof. Using the periodicity of the integrand, we estimate the error of cubature formula (5.6) as follows:

$$|R_{nn}|\leq\left|\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\int\int_{\Delta_{kl}}\left[\frac{f(\sigma_1,\sigma_2)-f(x'_i,x'_j)}{\left(sin^2\frac{\sigma_1-s_1}{2}+sin^2\frac{\sigma_2-s_2}{2}\right)^{\lambda}}-\right.\right.$$

$$\left.\left.-\frac{f(x'_k,x'_l)-f(x'_i,x'_j)}{(\left(sin^2\frac{\sigma_1-x'_i}{2}+sin^2\frac{\sigma_2-x'_j}{2}\right)^{\lambda}}\right]d\sigma_1 d\sigma_2\right|\leq$$

$$\leq \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int\int_{\Delta_{kl}} \frac{f(\sigma_1,\sigma_2) - f(x'_k, x'_l)}{\left(sin^2\frac{\sigma_1 - s_1}{2} + sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \int\int_{\Delta_{kl}} (f(x'_k, x'_l) - f(x'_i, x'_j)) \times \right.$$

$$\left. \times \left[\frac{1}{\left(sin^2\frac{\sigma_1 - s_1}{2} + sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} - \frac{1}{\left(sin^2\frac{\sigma_1 - x'_i}{2} + sin^2\frac{\sigma_2 - x'_j}{2}\right)^\lambda} \right] d\sigma_1 d\sigma_2 \right| =$$

$$= r_1 + r_2.$$

Let us estimate each of the sums r_1 and r_2 separately. One has:

$$r_1 \leq \left| \sum_{k=i-M}^{i+M} \sum_{l=j-M}^{j+M} \int\int_{\Delta_{kl}} \left[\frac{f(\sigma_1,\sigma_2) - f(x'_k, x'_l)}{\left(sin^2\frac{\sigma_1 - s_1}{2} + sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \sum_{k=0}^{n-1} {\sum_{l=0}^{n-1}}' \int\int_{\Delta_{kl}} \left[\frac{f(\sigma_1,\sigma_2) - f(x'_k, x'_l)}{\left(sin^2\frac{\sigma_1 - s_1}{2} + sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} d\sigma_1 d\sigma_2 \right| =$$

$$= r_{11} + r_{12},$$

where $\sum\sum'$ means summation over (k,l) such that
$\Delta_{kl} \notin \Delta^*$, $\Delta^* = [x_{i-M}, x_{i+M+1}; x_{j-M}, x_{j+M+1}], M = [lnn]$.

Furthermore

$$r_{11} \leq \frac{c}{n^\alpha} \int\int_{\Delta^*} \frac{d\sigma_1 d\sigma_2}{\left(sin^2\frac{\sigma_1 - s_1}{2} + sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} \leq$$

$$\leq \frac{c}{n^\alpha} \int_0^{2\pi M/n} \int_0^{2\pi} \frac{d\rho d\phi}{\rho^{2\lambda - 1}} \leq \frac{c \log n}{n^{\alpha + 2 - 2\lambda}} = o\left(\frac{1}{n^\alpha}\right).$$

Estimating r_{12}, one can assume without loss of generality $(i,j) = (0,0)$, and get:

$$r_{12} \leq 4 \int_0^{\pi/n} \int_0^{\pi/n} (\omega_1(\sigma_1) + \omega_2(\sigma_2)) d\sigma_1 d\sigma_2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq 4 \int_0^{\pi/n} \int_0^{\pi/n} (\sigma_1^\alpha + \sigma_2^\alpha) d\sigma_1 d\sigma_2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq \frac{8}{1+\alpha} \left(\frac{\pi}{n}\right)^{2+\alpha} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} h_{kl}(s_1, s_2, \sigma_1, \sigma_2) \leq$$

$$\leq \frac{1 + o(1)}{1+\alpha} 2 \left(\frac{\pi}{n}\right)^\alpha \int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{\left(sin^2\frac{\sigma_1}{2} + sin^2\frac{\sigma_2}{2}\right)^\lambda}.$$

Here

$$h_{kl}(s_1, s_2; \sigma_1, \sigma_2) = \sup_{(\sigma_1,\sigma_2)\in\Delta_{kl}} h(s_1, s_2; \sigma_1, \sigma_2).$$

Combining the estimates of r_{11} and r_{12}, one gets:

$$r_1 \le \frac{1+o(1)}{1+\alpha} 2 \left(\frac{\pi}{n}\right)^{\alpha} \gamma$$

Let us estimate r_2. To this end we estimate the difference

$$r_2(k,l) = \int\int_{\Delta_{kl}} \left| f(x'_k, x'_l) - f(x'_i, x'_j) \right| \times$$

$$\times \left| \left[\frac{1}{\left(sin^2 \frac{\sigma_1 - s_1}{2} + sin^2 \frac{\sigma_2 - s_2}{2}\right)^{\lambda}} - \frac{1}{\left(sin^2 \frac{\sigma_1 - x'_i}{2} + sin^2 \frac{\sigma_2 - x'_j}{2}\right)^{\lambda}} \right] d\sigma_1 d\sigma_2 \right|.$$

First, we estimate

$$r_2(i,j) \le \frac{c}{n^{\alpha}} \int\int_{\Delta_{ij}} \left| \frac{1}{\left(sin^2 \frac{\sigma_1 - s_1}{2} + sin^2 \frac{\sigma_2 - s_2}{2}\right)^{\lambda}} - \frac{1}{\left(sin^2 \frac{\sigma_1 - x'_i}{2} + sin^2 \frac{\sigma_2 - x'_j}{2}\right)^{\lambda}} \right| d\sigma_1 d\sigma_2 \le$$

$$\le \frac{c}{n^{2+\alpha-2\lambda}}.$$

The value $r_2(k,l)$ is estimated similarly for $|k-i| \le 3$ and $|l-j| \le 3$.

Let us estimate $r_2(k,l)$ for other values of k and l.

One has:

$$r_2(k,l) = \int\int_{\Delta_{kl}} \left| f(x'_k, x'_l) - f(x'_i, x'_j) \right| \times$$

$$\times \left| \frac{1}{\left(sin^2 \frac{\sigma_1 - s_1}{2} + sin^2 \frac{\sigma_2 - s_2}{2}\right)^{\lambda}} - \frac{1}{\left(sin^2 \frac{\sigma_1 - x'_i}{2} + sin^2 \frac{\sigma_2 - x'_j}{2}\right)^{\lambda}} \right| d\sigma_1 d\sigma_2 \le$$

$$\le \frac{c}{n} \int\int_{\Delta_{kl}} \left[|x'_k - x'_i|^{\alpha} + |x'_l - x'_j|^{\alpha} \right] \left[\left(\frac{|k-i|}{n} \right) + \left(\frac{|l-j|}{n} \right) \right] \times$$

$$\times \left| \frac{1}{\left(sin^2 \frac{\sigma_1 - x'_i + \theta_1(s_1 - x'_i)}{2} + sin^2 \frac{\sigma_2 - s_2}{2}\right)^{1+\lambda}} + \right.$$

$$\left. + \frac{1}{\left(sin^2 \frac{\sigma_1 - x'_i + \theta_1(s_1 - x'_i)}{2} + sin^2 \frac{\sigma_2 - x'_j + \theta_2(s_2 - x'_j)}{2}\right)^{1+\lambda}} \right| \le$$

$$\le \frac{c}{n^3} \left(\left| \frac{|k-i|}{n} \right|^{\alpha} + \left| \frac{|l-j|}{n} \right|^{\alpha} \right) \left(\left| \frac{|k-i|}{n} \right| + \left| \frac{|l-j|}{n} \right| \right) \left(\frac{n^2}{|k-i|^2 + |l-j|^2} \right)^{1+\lambda} \le$$

$$\le \frac{c}{n^{\alpha+2-2\lambda}} \frac{(|k-i| + |l-j|)^{1+\alpha}}{(|k-i|^2 + |l-j|^2)^{1+\lambda}} \le$$

$$\leq \frac{c}{n^{\alpha+2-2\lambda}} \frac{(|k-i|^2+|l-j|^2)^{(1+\alpha)/2}}{(|k-i|^2+|l-j|^2)^{1+\lambda}} \leq$$

$$\leq \frac{c}{n^{\alpha+2-2\lambda}} \frac{1}{(|k-i|^2+|l-j|^2)^{1/2-\alpha/2+\lambda}}.$$

To estimate r_2, one sums up the last expression over k and l. Without loss of generality assume $(i,j)=(0,0)$. Then

$$r_2 \leq \frac{c}{n^{\alpha+2-2\lambda}} \left(16 + 4 \sum_{k=0}^{[n/2]+1} \sum_{l=0}^{[n/2]+1}{}' \frac{1}{(k^2+l^2)^{\lambda+1/2-\alpha/2}} \right),$$

where $\sum\sum'$ means summation over k and l such that $k>3$ or $l>3$.

One has:

$$\sum_{k=0}^{[n/2]+1} \sum_{l=0}^{[n/2]+1}{}' \frac{1}{(k^2+l^2)^{\lambda+1/2-\alpha/2}} \leq$$

$$\leq A \left[\sum_{k=3}^{[n/l]+1} \frac{1}{k^{2\lambda+1-\alpha}} + \sum_{k=3}^{[n/2]+1} \sum_{l=3}^{[n/2]+1} \frac{1}{(k^2+l^2)^{\lambda+1/2-\alpha/2}} \right] \leq$$

$$\leq A \begin{cases} 1, & if \quad 2\lambda-\alpha>1; \\ \log n, & if \quad 2\lambda-\alpha=1; \\ n^{1-2\lambda+\alpha}, & if \quad 2\lambda-\alpha<1. \end{cases}$$

Hence

$$\mathbf{r_2} \leq A \begin{cases} n^{-(\alpha+2-2\lambda)}, & if \quad 2\lambda-\alpha>1; \\ n^{-1}\log n, & if \quad 2\lambda-\alpha=1; \\ n^{-1}, & if \quad 2\lambda-\alpha<1. \end{cases}$$

Thus, if $\alpha<1$, then

$$r_2 \leq o(n^{-\alpha}).$$

Combining the estimates of r_1 and r_2, one gets:

$$R_{nn}[\Psi] \leq \gamma \frac{(2+o(1))}{1+\alpha} \left(\frac{\pi}{n}\right)^{\alpha}.$$

Theorem 5.5 follows from the comparison of this inequality with the lower bound of the value $\zeta_{nn}[H_{\alpha,\alpha}(D)]$, mentioned in the Corollary to Theorem 5.1. ■

Remark. *If $\alpha=1$, the cubature formula (5.6) is optimal with respect to order.*

The proof of Theorem 5.5 yields also the following result:

Theorem 5.5'. *Let $\Psi=H_{\alpha\alpha}(D), 0<\alpha\leq 1$. Among all possible cubature formulas (5.1) with $\rho_1=\rho_2=0$, formula*

$$Kf = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(x'_k,x'_l) \int\int_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{\left(sin^2\left(\frac{\sigma-s_1}{2}\right)+sin^2\left(\frac{\sigma-s_2}{2}\right)\right)^{\lambda}} + R_{nn},$$

which has the error

$$R_{nn}[\Psi] = \frac{(2+o(1))\gamma}{1+\alpha}\left(\frac{\pi}{n}\right)^{\alpha},$$

is asymptotically optimal.

To apply formula (5.6), one has to calculate the integrals

$$I_{kl} = \int\int_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{\left(sin^2\frac{\sigma_1 - x'_i}{2} + sin^2\frac{\sigma_2 - x'_j}{2}\right)^{\lambda}} \tag{5.7}$$

for $k, l = 0, 1, \ldots, n-1$. Exact values of these integrals for arbitrary values λ are apparently unknown. Therefore the procedure of numerical calculation of integrals (5.7) should be given for practical application of formula (5.6).

Let $k = i$ and $l = j$. Then the integral I_{ij} is replaced by the integral

$$p_{ij}{}^{*} = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}}\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \frac{d\sigma_1 d\sigma_2}{\left(sin^2\frac{\sigma_1}{2} + sin^2\frac{\sigma_2}{2}\right)^{\lambda} + h}, \qquad h > 0,$$

which can be calculated by cubature formulas (in particular, Gauss quadrature rule) with arbitrary degree of accuracy because the function

$\frac{1}{\left(sin^2\frac{\sigma_1}{2} + sin^2\frac{\sigma_2}{2}\right)^{\lambda} + h}$,

has derivatives up to arbitrary order. The choice of parameter h is discussed in Section 8.

Let $k = i, l \neq j$, and

$$I_{il} = \frac{4\pi^2}{n^2}\left(sin^2\frac{x'_l - x'_j}{2}\right)^{-\lambda} = p^*_{il}.$$

Let $k \neq i$, $l = j$, and

$$I_{kj} = \frac{4\pi^2}{n^2}\left(sin^2\frac{x'_k - x'_i}{2}\right)^{-\lambda} = p^*_{kj}.$$

Let $k \neq i$, $l \neq j$, and

$$I_{kl} = \frac{4\pi^2}{n^2}\left(sin^2\frac{x'_k - x'_i}{2} + sin^2\frac{x'_l - x'_j}{2}\right)^{-\lambda} = p^*_{kl}.$$

The integral Kf is calculated by the formula

$$Kf = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} p^*_{kl} f(x'_k, x'_l) + R_{nn}(f, p^*_{kl}, x_k, y'_l). \tag{5.8}$$

Formula (5.8) is not optimal since it is not exact on constant functions $f(x, y) = const$. But one can estimate the error of this formula:

$$|R_{nn}(f, p^*_{kl}, x'_k, y'_l))| \leq M \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} |I_{kl} - p^*_{kl}| + R_{nn}(\Psi),$$

where $M = \max|f(x, y)|$.

The values $|I_{kl} - p^*_{kl}|$ are easily estimated, and one gets the conclusion of Theorem 5.5'.

Classes of smooth functions

Theorem 5.6. *Assume* $\varphi \in \tilde{W}^{r,r}(1)$. *Let* $\Psi = \tilde{W}^{r,r}(1)$, *and calculate the integral* $K\varphi$ *by formula (5.1) with* $\rho_1 = r-1$, $\rho_2 = r-1$, *and* $n_1 = n_2 = n$. *Then the cubature formula*

$$K\varphi = \int_0^{2\pi}\int_0^{2\pi} \frac{\varphi_{mn}(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2}{(sin^2(\sigma_1 - s_1)/2 + sin^2(\sigma_2 - s_2)/2)^\lambda} + R_{mn}(\varphi) \tag{5.9}$$

is asymptotically optimal.

Before proving Theorem 5.6, let us describe the construction of the spline φ_{mn}. Let $x_k = 2k\pi/n,\ k = 0,1,\dots,n$. Divide the sides of the squares $\Omega = [0,2\pi;0,2\pi]$ into n equal parts. Denote by Δ_{kl} the rectangle $\Delta_{kl} = [2k\pi/n, 2(k+1)\pi/n; 2l\pi/n, 2(l+1)\pi/n], k,l = 0,1,\dots,n-1$. Let $(s_1,s_2) \in \Delta_{ij}$. First we approximate $\varphi(\sigma_1,\sigma_2)$ as a function of σ_2, and construct a spline $\varphi_n(\sigma_1,\sigma_2)$ by the following rule. Let σ_1 be an arbitrary fixed number, $0 \le \sigma_1 \le 2\pi$. On the segments $[2k\pi/n, 2(k+1)\pi/n]$ for $k \neq j-2,\dots,j+1$, one has:

$$\varphi_n(\sigma_1,\sigma_2) = \sum_{l=0}^{r-1}\left[\frac{\varphi^{(0,l)}(\sigma_1, 2k\pi/n)}{l!}(\sigma_2 - 2k\pi/n)^l + B_l\delta^{(l)}(\sigma_1,(k+1)/n)\right],$$

where

$$\delta(\sigma_1,\sigma_2) := \varphi(\sigma_1,\sigma_2) - \sum_{l=0}^{r-1}\frac{\varphi^{(0,l)}(\sigma_1,2k\pi/n)}{l!}(\sigma_2 - 2k\pi/n)^l.$$

The coefficients B_l are defined by the equation

$$(2(k+1)\pi/n - \sigma_2)^r - \sum_{l=0}^{r-1}\frac{B_l r!}{(r-l-1)!}\frac{2\pi}{n}\left(2\pi(k+1)/n - \sigma_2\right)^{r-l-1} =$$
$$= (-1)^r R_{r1}\left(2\pi(2k+1)/2n; \pi/n; \sigma_2\right),$$

where $R_{r1}(a,h,x)$ is a polynomial of degree r, least deviating from zero in the norm of the space L on the segment $[a-h, a+h]$. On the segment $[2\pi(j-2)/n, 2\pi(j+2)/n]$ the function $\varphi_n(\sigma_1,\sigma_2)$ is defined by the partial sum of the Taylor series:

$$\varphi_n(\sigma_1,\sigma_2) = \varphi(\sigma_1, 2\pi j/n) + \frac{\varphi^{(0,1)}(\sigma_1,2\pi j/n)}{1!}(\sigma_2 - j/n) + \dots +$$
$$+\frac{\varphi^{(0,r-1)}(\sigma_1,2\pi j/n)}{(r-1)!}(\sigma_2 - 2\pi j/n)^{r-1}.$$

We define the function $\varphi_{nn}(\sigma_1,\sigma_2)$ by analogy with the function $\varphi_n(\sigma_1,\sigma_2)$.

Proof of Theorem 5.6. Let $(s_1,s_2) \in \Delta_{ij}$. The error of formula (5.9) we estimate by the inequality

$$|R_{nn}| \leqslant \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}' \left|\iint_{\Delta_{kl}} \frac{\varphi(\sigma_1,\sigma_2) - \varphi_{nn}(\sigma_1,\sigma_2)}{\left(\sin^2\frac{\sigma_1 - s_1}{2} + \sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} d\sigma_1 d\sigma_2\right| +$$
$$+\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}'' \left|\int_{\Delta_{kl}} \frac{\varphi(\sigma_1,\sigma_2) - \varphi_{nn}(\sigma_1,\sigma_2)}{\left(\sin^2\frac{\sigma_1 - s_1}{2} + \sin^2\frac{\sigma_2 - s_2}{2}\right)^\lambda} d\sigma_1 d\sigma_2\right| = r_1 + r_2, \tag{5.10}$$

where $\sum\limits_{k,l}{}'$ means summation over (k,l) such that $i-1 \leqslant k \leqslant i+1$, $0 \leqslant l \leqslant n-1$ or $0 \leqslant k \leqslant n-1$, $j-1 \leqslant l \leqslant j+1$, and $\sum\limits_{k,l}{}''$ means summation over the other values of (k,l).

Let us estimate each of the sums r_1 and r_2 separately. In addition without loss of generality assume that $\iint\limits_{\Delta_{kl}} (\varphi(\sigma_1,\sigma_2)-\varphi_{nn}(\sigma_1,\sigma_2))d\sigma_1 d\sigma_2 \geqslant 0$. Then

$$r_1 \leqslant \sum_{k=0}^{n-1} {\sum_{l=0}^{n-1}}' |\varphi(\sigma_1,\sigma_2)-\varphi_{nn}(\sigma_1,\sigma_2)| \iint\limits_{\Delta_{kl}} \frac{d\sigma_1 d\sigma_2}{\left(\sin^2\frac{\sigma_1-s_1}{2}+\sin^2\frac{\sigma_2-s_2}{2}\right)^\lambda} \leqslant$$

$$\leqslant A \begin{cases} n^{-(r+1)} & ,\lambda \leq 1/2 \\ n^{-(r+2-2\lambda)} & ,\lambda > 1/2; \end{cases} \tag{5.11}$$

$$r_2 \leqslant 4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \frac{1}{\left(\sin^2\frac{x_k-s_1}{2}+\sin^2\frac{x_l-s_2}{2}\right)^\lambda} \iint\limits_{\Delta_{kl}} \psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 -$$

$$-4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \left[\frac{1}{\left(\sin^2\frac{x_k-s_1}{2}+\sin^2\frac{x_l-s_2}{2}\right)^\lambda} - \frac{1}{\left(\sin^2\frac{x_{k+1}-s_1}{2}+\sin^2\frac{x_{l+1}-s_2}{2}\right)^\lambda}\right].$$

$$\cdot \iint\limits_{\Delta_{kl}} \psi^-(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 = r_{21}+r_{22}, \tag{5.12}$$

where $\psi(\sigma_1,\sigma_2)=\varphi(\sigma_1,\sigma_2)-\varphi_{nn}(\sigma_1,\sigma_2)$,

$$\psi^+(\sigma_1,\sigma_2) = \begin{cases} \psi(\sigma_1,\sigma_2) & , \text{ if } \psi(\sigma_1,\sigma_2) \geqslant 0 \\ 0 & , \text{ if } \psi(\sigma_1,\sigma_2) < 0; \end{cases}$$

$$\psi^-(\sigma_1,\sigma_2) = \begin{cases} 0 & , \text{ if } \psi(\sigma_1,\sigma_2) \geqslant 0; \\ -\psi(\sigma_1,\sigma_2) & , \text{ if } \psi(\sigma_1,\sigma_2) < 0. \end{cases}$$

It is obvious

$$4 \sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \frac{1}{\left(\sin^2\frac{x_k-s_1}{2}+\sin^2\frac{x_l-s_2}{2}\right)^\lambda} \leqslant \frac{1+o(1)}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{\left(\sin^2\frac{\sigma_1}{2}+\sin^2\frac{\sigma_2}{2}\right)^\lambda} \tag{5.13}$$

Let us estimate the integral

$$i = \iint\limits_{\Delta_{kl}} \psi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2 \leqslant \left| \iint\limits_{\Delta_{kl}} \Big(\varphi(\sigma_1,\sigma_2)-\varphi_n(\sigma_1,\sigma_2)\Big)d\sigma_1 d\sigma_2 \right| +$$

$$+ \left| \iint\limits_{\Delta_{kl}} \Big(\varphi_n(\sigma_1,\sigma_2)-\varphi_{nn}(\sigma_1,\sigma_2)\Big)d\sigma_1 d\sigma_2 \right| = i_1+i_2. \tag{5.14}$$

Since the expressions i_1 and i_2 are estimated similarly, we estimate only i_1. One has:

$$i_1 \leqslant \frac{2\pi}{n} \max_{s_1} \left| \int_{x_l}^{x_{l+1}} \Big(\varphi(s_1,\sigma_2) - \varphi_n(s_1,\sigma_2)\Big) d\sigma_2 \right|.$$

This integral is a continuous function of s_1, which attains its maximum at a point s^*, and

$$i_1 \leqslant \frac{2\pi}{n} \left| \int_{x_l}^{x_{l+1}} \Big(\varphi(s^*,\sigma_2) - \varphi_n(s^*,\sigma_2)\Big) d\sigma_2 \right| \leqslant$$

$$\leqslant \frac{2\pi}{r!n} \int_{x_l}^{x_{l+1}} \big|\varphi^{(0,r)}(s^*,\sigma_2)\big| \Bigg| (x_{l+1}-\sigma_2)^r -$$

$$- \sum_{j=0}^{r-1} \frac{B_{lj}(x_{l+1}-x_l) r!}{(r-1-j)!} (x_{l+1}-\sigma_2)^{r-j-1} \Bigg| d\sigma_2 \leqslant$$

$$\leqslant \frac{2\pi}{r!n} \int_{x_l}^{x_{l+1}} \left| (x_{l+1}-\sigma_2)^r - \sum_{j=0}^{r-1} \frac{B_{lj}(x_{l+1}-x_l) r!}{(r-1-j)!} (x_{l+1}-\sigma_2)^{r-j-1} \right| d\sigma_2 =$$

$$= \frac{2\pi}{r!n} \int_{x_l}^{x_{l+1}} \big|R_{r1}(\sigma_2)\big| d\sigma_2 \leqslant \frac{4}{(r+1)!} \left(\frac{\pi}{n}\right)^{r+2} R_{r1}(1). \tag{5.15}$$

From inequalities (5.14) and (5.15) one gets:

$$i \leqslant \frac{8}{(r+1)!} \left(\frac{\pi}{n}\right)^{r+2} R_{r1}(1)$$

and

$$r_{21} \leqslant \frac{2+o(1)}{(r+1)!} \left(\frac{\pi}{n}\right)^{r} R_{r1}(1) \int_0^{2\pi}\int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{\left(\sin^2\frac{\sigma_1}{2} + \sin^2\frac{\sigma_2}{2}\right)^{\lambda}}. \tag{5.16}$$

One has:

$$r_{22} = o(n^{-r}). \tag{5.17}$$

Estimate (5.17) follows from the inequalities:

$$\left| \iint_{\Delta_{kl}} \psi^-(\sigma_1,\sigma_2) d\sigma_1 d\sigma_2 \right| \leqslant \iint_{\Delta_{kl}} \big|\psi(\sigma_1,\sigma_2)\big| d\sigma_1 d\sigma_2 = O(n^{-r-2})$$

and

$$\sum_{k=i+2}^{i+1+[(n-1)/2]} \sum_{l=j+2}^{j+1+[(n-1)/2]} \left| \frac{1}{\left(\sin^2\frac{x_k-s_1}{2} + \sin^2\frac{x_l-s_2}{2}\right)^{\lambda}} - \frac{1}{\left(\sin^2\frac{x_{k+1}-s_1}{2} + \sin^2\frac{x_{l+1}-s_2}{2}\right)^{\lambda}} \right| \leqslant$$

$$\leqslant An^{2\lambda} \sum_k \sum_l \frac{(k-i)+(l-j)}{\big((k-i)^2+(l-j)^2\big)^{\lambda+1}} \leqslant c \begin{cases} n & ,\lambda < 1/2 \\ n\log n & ,\lambda = 1/2 \\ n^{2\lambda} & ,\lambda > 1/2. \end{cases}$$

The estimate
$$R_{nn}(\Psi) \le (1+o(1))\frac{2\pi^r R_{r1}(1)}{(r+1)!(n-1+[R_{r1}(1)]^{1/r})^r}\gamma$$
follows from inequalities (5.10), (5.11), (5.16), and (5.17).

Theorem 5.5 follows from the comparison of the values $\zeta_{nn}[\Psi]$ and $R_{nn}[\Psi]$. ■

Let us construct cubature formulas for calculating integrals Kf on classes of functions $W^{rr}(1)$. These formulas will be less accurate than the ones in Theorem 5.3, but they will be optimal with respect to order, and easier to apply.

First, we investigate the smooth function
$$\psi(t_1,t_2) = \int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{f(\tau_1,\tau_2)d\tau_1 d\tau_2}{\left(\sin^2\frac{\tau_1-t_1}{2}+\sin^2\frac{\tau_2-t_2}{2}\right)^\lambda}$$
assuming $f(t_1,t_2)\in \tilde{W}^{r,r}$. Changing the variables $\tau_1=\tau_1-t,\ \tau_2=\tau_2-t$, in the last integral, one gets:
$$\psi(t_1,t_2) = \int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{f(\tau_1+t_1,\tau_2+t_2)d\tau_1 d\tau_2}{\left(\sin^2\frac{\tau_1}{2}+\sin^2\frac{\tau_2}{2}\right)^\lambda}$$

Thus, $\psi(t_1,t_2)\in W^{r,r}$.

Remark. *It is known [9] that Kolmogorov and Babenko widths on the class of functions $W^{r,r}(1)$ are equal to $\delta_n(W^{r,r}(1)) \asymp d_n(W^{r,r}(1),C) \asymp \frac{1}{n^{r/2}}$. Hence the recovery of the function $\psi(t_1,t_2)$ using n functionals is not possible with accuracy greater than $O\left(\frac{1}{n^{r/2}}\right)$. More precise conclusions are obtained in Theorems 5.3 and 5.4.*

Thus, for recovery of a function $\psi(t_1,t_2),\ (t_1,t_2)\in[0,2\pi]^2$ with the accuracy $O(n^{-r/2})$, it is sufficient to calculate the value of the function $\psi(t_1,t_2)$ at the nodes (v_k,v_l), where $v_k=2k\pi/N$, $k,l=0,1,\dots,N$, and $N^2=n$, and to use the local spline $\psi_N(t_1,t_2)$ of degree r with respect to each variable.

Let us describe the construction of such spline.

Assume for simplicity that $M:=N/r$ is an integer, and cover the domain $[0,2\pi]^2$ with the squares $\Delta_{kl}=[w_k,w_l],\ k,l=0,1,\dots,M-1$, here $w_k=2k\pi/M,\ k=0,\dots,M$. Approximate the function $\psi(t_1,t_2)$ in each domain Δ_{kl} by the interpolation polynomial $\psi_N(t_1,t_2,\Delta_{kl})$ constructed on the nodes $(x_i^k,x_j^l),\ i,j=0,1,\dots,r,\ x_i^k=w_k+\frac{2\pi}{Mr}i,\ i=0,1,\dots,r.$

Denote the local spline, which is defined by the polynomials $\psi_N(t_1,t_2,\Delta_{kl})$, by $\psi_N(t_1,t_2)$.

If the values $\psi(v_k,v_l)$ are calculated by formula (5.9) with the accuracy $O(n^{-r/2})$, then
$$\|\psi(t_1,t_2)-\psi_N(t_1,t_2)\|_C \le O(n^{-r/2}).$$

Therefore the spline $\psi_N(t_1,t_2)$ is optimal with respect to order, and a method for recovery of the function $\psi(t_1,t_2)$, which has the error $O(n^{-r/2})$ (in the sup −norm) is constructed.

6. Optimal methods for calculating integrals of the form Tf.

Lower bounds for the functionals ζ_{mn} and ζ_N.

First we get a lower bound for the error of formula (2.1) with $\rho_1=\rho_2=0$ and $n_1=n_2=n$, on Hölder classes.

Theorem 6.1. *Let $\Psi = H_{\alpha\alpha}(D)$, and calculate the integral Tf by formula (2.1) with $n_1=n_2=n$ and $\rho_1=\rho_2=0$. Then the estimate:*

$$\zeta_{nn}[\Psi] \geq \frac{(1+o(1))}{2^{2\lambda}(1+\alpha)n^\alpha} \int_{-1}^{1}\int_{-1}^{1} \frac{dt_1 dt_2}{(\tau_1^2+t_1^2)^\lambda} \tag{6.1}$$

holds.

Proof. Let $n>0$ be an integer, $L=[n/\log n]$. Let $v_k := -1+2k/L$, $k=0,1,\dots,L$. By (ξ_k,η_l) we denote a set which is the union of nodes (x_i,y_j), $i,j=1,2,\dots,n$ of formula (2.1) and the nodes (v_i,v_j), $i,j=1,2,\dots,L$. Let $\Delta_{kl}=[v_k,v_{k+1};v_l,v_{l+1}]$, $k,l=0,1,\dots,L-1$. Let $0\leq \psi(t_1,t_2)\in H_{\alpha\alpha}(D)$, where $D=[-1,1]^2$, vanishing at the nodes (ξ_k,η_l), $k,l=0,1,\dots,N$, where $N=n+L$.

Consider the integral

$$(T\psi)(v_i,v_j) = \int_{-1}^{1}\int_{-1}^{1} \frac{\psi(\tau_1,\tau_2)d\tau_1 d\tau_2}{((\tau_1-v_i)^2+(\tau_2-v_j)^2)^\lambda} =$$

$$= \left(\sum_{k=i}^{L-1}\sum_{l=j}^{L-1} + \sum_{k=i}^{L-1}\sum_{l=0}^{j-1} + \sum_{k=0}^{i-1}\sum_{l=j}^{L-1} + \sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\right) \times$$

$$\times \int\int_{\Delta_{kl}} \frac{\psi(\tau_1,\tau_2)d\tau_1 d\tau_2}{((\tau_1-v_i)^2+(\tau_2-v_j)^2)^\lambda} \geq$$

$$\geq \sum_{k=0}^{L-i-1}\sum_{l=0}^{L-j-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{k+i,l+j}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$+ \sum_{k=0}^{L-i-1}\sum_{l=0}^{j-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{k+i,j-l-1}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$+ \sum_{k=0}^{i-1}\sum_{l=0}^{L-j-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{i-k-1,j+l}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$+ \sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{i-k-1,j-l-1}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 =$$

$$= \sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{U(L-i-1-k)U(L-j-1-l)}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{k+i,l+j}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$+ \sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{U(L-i-1-k)U(j-1-l)}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{k+i,j-l-1}} \psi(\tau_1,\tau_2)d\tau_1 d\tau_2 +$$

$$+\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{U(i-1-k)U(L-j-1-l)}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{i-k-1,j+l}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+$$

$$+\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\left(\frac{L}{2}\right)^{2\lambda}\frac{U(i-1-k)U(j-1-l)}{((k+1)^2+(l+1)^2)^\lambda}\int\int_{\Delta_{i-k-1,j-l-1}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

Here $U(k)=1$ for $k\geq 0$, and $U(k)=0$ for $k<0$.

Averaging the above inequality over all i and j, $i,j=0,1,\dots,L-1$, one gets:

$$R_{nn}(\Psi,p_{kl};x_k,y_l)\geq\frac{1}{L^2}\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}T(\psi)(\xi_i,\eta_j)\geq$$

$$\geq\frac{1}{L^{2-2\lambda}2^{2\lambda}}\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\left[\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}U(L-i-1-k)\times\right.$$

$$\times U(L-j-1-l)\int\int_{\Delta_{k+i,l+j}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+$$

$$+\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}U(L-i-1-k)U(j-1-l)\int\int_{\Delta_{k+i,j-l-1}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+$$

$$+\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}U(i-1-k)U(L-j-1-l)\int\int_{\Delta_{i-k-1,j+l}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+$$

$$\left.+\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}U(i-1-k)U(j-1-l)\int\int_{\Delta_{i-k-1,j-l-1}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2\right]\geq$$

$$\geq\frac{1}{L^{2-2\lambda}2^{2\lambda}}\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\left[\int_{v_k}^{1}\int_{v_l}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+\right.$$

$$\left.+\int_{v_k}^{1}\int_{-1}^{v_{L-l-2}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+\int_{-1}^{v_{L-k-2}}\int_{v_l}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2+\int_{-1}^{v_{L-k-2}}\int_{-1}^{v_{L-l-2}}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2\right]\geq$$

$$\geq\frac{1}{L^{2-2\lambda}2^{2\lambda}}\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\frac{1}{((k+1)^2+(l+1)^2)^\lambda}\int_{-1}^{1}\int_{-1}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2. \qquad (6.2)$$

From inequality (6.2) it follows that

$$\zeta_{nn}[H_{\alpha\alpha}(D)]\geq(1+o(1))\frac{1}{L^{2-2\lambda}2^{2\lambda}}\sum_{k=1}^{L-1}\sum_{l=1}^{L-1}\frac{1}{(k^2+l^2)^\lambda}\int_{-1}^{1}\int_{-1}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2=$$

$$=\frac{1+o(1)}{2^{2\lambda}4}\int_{-1}^{1}\int_{-1}^{1}\frac{dt_1 dt_2}{(t_1^2+t_2^2)^\lambda}\int_{-1}^{1}\int_{-1}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2. \qquad (6.3)$$

From Theorem 4.2 and Lemma 4.4 it follows that the inequality

$$\int_{-1}^{1}\int_{-1}^{1}\psi(\tau_1,\tau_2)d\tau_1 d\tau_2 \geq \frac{4}{1+\alpha}\frac{1}{n^\alpha} \tag{6.4}$$

is valid for an arbitrary vector of the weights and the nodes (X, Y, P) on the class $H_{\alpha\alpha}(D)$.

Theorem 6.1 follows from inequalities (6.3) and (6.4). ■

Theorem 6.2. *Let $\Psi = C_2^r(1)$, and calculate the integral Tf by formula (2.1) with $\rho_1 = \rho_2 = 0$. If $n_1 = n_2 = n$, then*

$$\zeta_{nn}[\Psi] \geq (1+o(1))\frac{2K_r}{2^{2\lambda}(\pi n)^r}\int_{-1}^{1}\int_{-1}^{1}\frac{ds_1 ds_2}{(s_1^2)+s_2^2))^\lambda},$$

where K_r is the Favard constant.

Proof. Let

$$\psi(s_1,s_2) = \psi_1(s_1) + \psi_2(s_2),$$

where $0 \leq \psi_1(s) \in W^r(1)$, vanishes at the nodes x_k, $k = 1, 2, \dots, n$, and $0 \leq \psi_2(s) \in W^r(1)$ vanishes at the nodes y_k, $k = 1, 2, \dots, n$.

For arbitrary nodes x_k, $k = 1, 2, \dots, n$, one has (see [11]):

$$\int_{-1}^{1}\psi_i(s)ds \geq \frac{2K_r}{(\pi n)^r},\ \ i = 1, 2.$$

Thus the inequality

$$\int_{-1}^{1}\int_{-1}^{1}\psi(s_1,s_2)ds_1 ds_2 \geq \frac{8K_r}{(\pi n)^r}$$

holds for arbitrary nodes $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$.

Theorem 6.2 follows from this estimate and inequality (6.3). ■

Theorem 6.3. *Let $\Psi = W_p^{r,r}(1)$, $r = 1, 2, \dots$, $1 \leq p \leq \infty$, and calculate the integral Tf by formula (2.1) with $\rho_1 = \rho_2 = r-1$ and $n_1 = n_2 = n$. Then the estimate*

$$\zeta_{nn}[\Psi] \geq (1+o(1))\frac{2^{1/q}R_{rq}(1)}{2^{2\lambda}r!(rq+1)^{1/q}(n-1+[R_{rq}(1)]^{1/r})^r}\int_{-1}^{1}\int_{-1}^{1}\frac{ds_1 ds_2}{(s_1^2+s_2^2)^\lambda}, \tag{6.5}$$

holds, where $R_{rq}(t)$ is a polynomial of degree r, least deviating from zero in $L_q([-1,1])$.

Proof. Let $L = [n/\log n]$. Consider the nodes (v_k, v_l), $v_k = \frac{2k}{L}$, $k, l = 0, 1, \dots, L-1$. By (ξ_i, η_j), $i, j = 0, 1, \dots, N-1$, $N = n + L$ denote the union of the nodes (x_k, y_l) and (ξ_i, ξ_j). Let $\psi(s_1, s_2) = \psi_1(s_1) + \psi_2(s_2)$, where $0 \leq \psi_1(s) \in W_p^r(1)$ vanishes with its derivatives up to order $r-1$ at the nodes ξ_i, $i = 0, 1, \dots, N-1$, and $0 \leq \psi_2(s) \in W_p^r(1)$ vanishes with its derivatives up to order $r-1$ at the nodes η_j, $j = 0, 1, \dots, N-1$. Assume that $\int\limits_{v_i}^{v_{i+1}}\psi_1(s)ds > 0$, $i = 0, 1, \dots, N-1$, and $\int\limits_{v_j}^{v_{j+1}}\psi_2(s)ds > 0$, $j = 0, 1, \dots, N-1$.

Using the argument similar to the one in the proof of Theorem 6.1, one gets:

$$\zeta_{nn}(\Psi, p_{kl}; v_k, v_l) \geq \frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} T(\psi)(v_i, v_j) \geq$$

$$\geq \frac{1}{L^{2-2\lambda} 2^{2\lambda}} \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} \frac{1}{((k+1)^2 + (l+1)^2)^\lambda} \int_{-1}^{1} \int_{-1}^{1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 =$$

$$= \frac{1 + o(1)}{2^{2\lambda} 4} \int_{-1}^{1} \int_{-1}^{1} \frac{dt_1 dt_2}{(t_1^2 + t_2^2)^\lambda} \int_{-1}^{1} \int_{-1}^{1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2. \tag{6.6}$$

From Theorem 4.1 and lemma 4.4 it follows that the inequality

$$\int_{-1}^{1} \int_{-1}^{1} \psi(\tau_1, \tau_2) d\tau_1 d\tau_2 \geq (1 + o(1)) \frac{2^{2+1/q} R_{rq}(1)}{r!(rq+1)^{1/q}(n - 1 + [R_{rq}(1)]^{1/q})^r} \tag{6.7}$$

is valid for arbitrary weights and the nodes (X, Y, P) on the class $H_{\alpha\alpha}(D)$.

Theorem 6.3 follows from inequalities (6.6)- (6.7). ■

Cubature formulas.

Let us construct a cubature formula for calculating the integral Tf on the Hölder class $H_{\alpha\alpha}(D)$. Let $x_k := -1 + 2k/n$, $k = 0, 1, \ldots, n$, $x'_k = (x_{k+1} + x_k)/2$, $k = 0, 1, \ldots, n-1$, and $\Delta_{kl} = [x_k, x_{k+1}; x_l, x_{l+1}]$, $k, l = 0, 1, \ldots, n-1$.

Calculate the integral Tf by the formula

$$Tf = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(x'_k, x'_l) \int\int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + R_{nn}(f). \tag{6.8}$$

Consider another cubature formula for calculating the integral Tf.

Let $(t_1, t_2) \in \Delta_{ij}$. By Δ_* denote the union of the square Δ_{ij} and of those squares Δ_{kl} which have common points with the Δ_{ij}. Consider the formula

$$Tf = f(x'_i, x'_j) \int\int_{\Delta_*} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} +$$

$$+ \sum_{k=0}^{n-1} \sum_{l=0}^{n-1}{}' f(x'_k, x'_l) \int\int_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda} + R_{nn}(f), \tag{6.9}$$

where $\sum\sum'$ means summation over the squares which do not belong to Δ_*.

Theorem 6.4. *Among all cubature formulas (2.1) with $\rho_1 = \rho_2 = 0$ and $n_1 = n_2 = n$, formula (6.8), with the error estimate (6.15), is optimal with respect to order.*

Remark. *Similar statement holds for formula (6.9).*

Proof of Theorem 6.4. Let us estimate errors of formulas (6.8) and (6.9).

The error of formula (6.8) can be estimated as follows:

$$|R_{nn}(f)| \le \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}' \int\limits_{\Delta_{kl}}\!\!\int \frac{|f(\tau_1,\tau_2)-f(x_i',x_j')|}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda}d\tau_1 d\tau_2 +$$

$$+\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}'' \int\limits_{\Delta_{kl}}\!\!\int \frac{|f(\tau_1,\tau_2)-f(x_k',x_l')|}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda}d\tau_1 d\tau_2 = r_1 + r_2, \tag{6.10}$$

where $\sum\sum'$ means summation over k and l such that the squares Δ_{kl} belong to Δ_*, and $\sum\sum''$ means summation over the other squares.

Let us estimate r_1 and r_2:

$$r_1 \le \frac{2}{n^\alpha}\int\limits_{\Delta_*}\!\!\int \frac{d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} \le \frac{c}{n^{2-2\lambda+\alpha}} = o(n^{-\alpha}); \tag{6.11}$$

$$r_2 \le \frac{4}{1+\alpha}\frac{1}{n^{2+\alpha}}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}'' h(\Delta_{kl}). \tag{6.12}$$

Here $h(\Delta_{kl})$ denotes the maximum value of the function $((\tau_1-t_1)^2+(\tau_2-t_2)^2)^{-\lambda}$ in the square Δ_{kl}.

One has:

$$\left|\int\limits_{\Delta_{kl}}\!\!\int \left[\frac{1}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} - h(\Delta_{kl})\right] d\tau_1 d\tau_2\right| =$$

$$= \left|\int\limits_{\Delta_{kl}}\!\!\int \left[\frac{1}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} - \frac{1}{((x_k-t_1)^2+(x_l-t_2)^2)^\lambda}\right] d\tau_1 d\tau_2\right| \le$$

$$\le \int\limits_{\Delta_{kl}}\!\!\int \left|\frac{2\lambda(x_k-t_1+q_1(\tau_1-x_k))(\tau_1-x_k))}{((x_k-t_1+q_1(\tau_1-x_k))^2+(\tau_2-t_2)^2)^{\lambda+1}}d\tau_1 d\tau_2\right| +$$

$$+\int\limits_{\Delta_{kl}}\!\!\int \left|\frac{2\lambda(x_l-t_2+q_2(\tau_2-x_l))(\tau_2-x_l))}{((x_k-t_1)+(x_l-t_2+q_2(\tau_2-x_l))^2)^{\lambda}}d\tau_1 d\tau_2\right| \le$$

$$\le \int\limits_{\Delta_{kl}}\!\!\int \frac{2\lambda(\tau_1-x_k)}{((x_k-t_1+q(\tau_1-x_k))^2+(\tau_2-t_2)^2)^{\lambda+1/2}}d\tau_1 d\tau_2\Bigg| +$$

$$+\int\limits_{\Delta_{kl}}\!\!\int \frac{2\lambda(\tau_2-x_l))}{((\tau_1-t_1)^2+(x_l-t_2+q_2(\tau_2-x_l))^2)^{\lambda+1/2}}d\tau_1 d\tau_2\Bigg| \le$$

$$\le \frac{2^4\lambda}{n^3}\frac{n^{2\lambda+1}}{(k^2+l^2)^{\lambda+1/2}} = \frac{2^4\lambda}{n^{2-2\lambda}}\frac{1}{(k^2+l^2)^{\lambda+1/2}},$$

where it was assumed that $k \ge i+1$, and $l \ge j+1$. Estimates for the other combinations of k and l are similar. Thus:

$$r_2 \le \frac{1}{(1+\alpha)}\frac{1}{n^\alpha}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}'' \int\limits_{\Delta_{kl}}\!\!\int \frac{d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} +$$

$$+\frac{1}{(1+\alpha)}\frac{2^3\lambda}{n^{2-2\lambda+\alpha}}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}''\frac{1}{(k^2+l^2)^{\lambda+1/2}}. \tag{6.13}$$

Let us estimate the last term in the above inequality.

One has:

$$\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}''\frac{1}{(k^2+l^2)^{\lambda+1/2}} \le \sum_{k=-[n/2]}^{n/2}\sum_{l=-[n/2]}^{n/2}{}^{*}\frac{1}{(k^2+l^2)^{\lambda+1/2}} \le$$

$$\le \mathbf{c}\begin{cases} 1, & \lambda > 1/2 \\ \log n, & \lambda = 1/2 \\ n^{1-2\lambda}, & \lambda < 1/2 \end{cases} \tag{6.14}$$

where $\sum\sum^*$ means summation over k and l, $(k,l) \neq (0,0)$.

In deriving (6.14) we have used the known result ([17], Theorem 56) which says that a number of points with integer-value coordinates, situated in the circle $x^2+y^2=r^2$, is equal to $\pi r^2 + O(r)$.

From inequalities (6.13) and (6.14) it follows that

$$r_2 \le \frac{(1+o(1))}{(1+\alpha)}\frac{1}{n^\alpha}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}{}''\int\int_{\Delta_{kl}}\frac{d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda}.$$

This and (6.11) yield:

$$R_{nn}[H_{\alpha\alpha}(D)] \le \frac{1+o(1)}{(1+\alpha)n^\alpha}\sup_{(t_1,t_2)\in D}\int_{-1}^{1}\int_{-1}^{1}\frac{d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} \le$$

$$\le \frac{1+o(1)}{(1+\alpha)n^\alpha}\int_{-1}^{1}\int_{-1}^{1}\frac{d\tau_1 d\tau_2}{(\tau_1^2+\tau_2^2)^\lambda}. \tag{6.15}$$

Theorem 6.4 follows from a comparison the estimates of $\zeta[H_{\alpha,\alpha}(D)]$ and $R_{nn}[H_{\alpha,\alpha}(D)]$. ■

Let us construct optimal with respect to order cubature formula for calculating integrals Tf on the classes W^{rr}. In the derivation of formula (5.9) the local spline $\varphi_n(t_1,t_2)$, approximating the function $\varphi(t_1,t_2)$ in the domain $[0,2\pi;0,2\pi]$, was constructed. A spline $f_{nn}(t_1,t_2)$, approximating the function $f(t_1,t_2)$ in the domain $[-1,1]\times[-1,1]$, can be constructed analogously. Calculate the integral Tf by the formula

$$Tf = \int_{-1}^{1}\int_{-1}^{1}\frac{f_{nn}(\tau_1,\tau_2)d\tau_1 d\tau_2}{((\tau_1-t_1)^2+(\tau_2-t_2)^2)^\lambda} + R_{nn}(f). \tag{6.16}$$

Theorem 6.5. *Let $\Psi = W^{r,r}(1), r=1,2,\ldots,$ and calculate the integral Tf by formula (2.1) with $\rho_1=\rho_2=r-1$, and $n_1=n_2=n$. Then cubature formula (6.16), which has the error*

$$R_{nn}(\Psi) \le (1+o(1))\frac{2R_{r1}(1)}{(r+1)!(n-1+[R_{r1}(1)]^{1/r})^r}\int_{-1}^{1}\int_{-1}^{1}\frac{d\tau_1 d\tau_2}{(\tau_1^2+\tau_2^2)^\lambda},$$

is optimal with respect to order. Here $R_{rq}(t)$ is a polynomial of degree r, least deviating from zero in $L_q([-1,1])$.

As in the proof of the Theorem 5.6 one gets the following estimate

$$R_{nn}(\Psi) \le (1+o(1))\frac{2R_{r1}(1)}{(r+1)!(n-1+[R_{r1}(1)]^{1/r})^r}\int\limits_{-1}^{1}\int\limits_{-1}^{1}\frac{d\tau_1 d\tau_2}{(\tau_1^2+\tau_2^2)^\lambda}.$$

Comparing this estimate with the estimate of $\zeta_{nn}[W^{r,r}(1)]$ from Theorem 6.3 one finishes the proof. ■

7. Calculation of weakly singular integrals on piecewise continuous surfaces.

In Sections 5 and 6 asymptotically optimal methods for calculating weakly singular integrals defined on the squares $[0,2\pi]^2$ or $[-1,1]^2$ were constructed.

It is of interest to study optimal methods for calculating weakly singular integrals on piecewise-Lyapunov surfaces.

Consider the integral

$$Jf = \iint\limits_G \frac{f(\tau_1,\tau_2,\tau_3)dS}{\Big((\tau_1-t_1)^2+(\tau_2-t_2)^2+(\tau_3-t_3)^2\Big)^\lambda},\ t_1,t_2,t_3\in G, \tag{7.1}$$

where G is a Lyapunov surface of class $L_s(B,\alpha)$.

We show that the results derived in Sections 5 and 6 can be partially generalized to the integrals (7.1).

Calculate integrals (7.1) by the formula:

$$Jf = \sum_{k=1}^{n}\sum_{|v|=0}^{\rho} p_{kv} f^{(v)}(M_k) + R_n(f,G,M_k,p_{kv},t), \tag{7.2}$$

where $t=(t_1,t_2,t_3),\ v=(v_1,v_2,v_3),\ |v|=v_1+v_2+v_3,\ f^{(v)}(t_1,t_2,t_3)=\frac{\partial^{|v|}f}{\partial t_1^{v_1}\partial t_2^{v_2}\partial t_3^{v_3}}$.

The error of formula (7.2) is:

$$R_n(f,G,M_k,p_{kv}) = \sup_{t\in G}|R_n(f,G,M_k,p_{kv},t)|.$$

Assume $f\in\Psi_1$, and $G\in\Psi_2$. Then the error of formula (7.2) on the classes Ψ_1 and Ψ_2 is:

$$R_n(\Psi_1,\Psi_2,M_k,P_{kv}) = \sup_{f\in\Psi_1,G\in\Psi_2} R_n(f,G,M_k,p_{kv}).$$

Let

$$\zeta_n[\Psi_1,\Psi_2] := \inf_{M_k,p_{kv}} R_n(\Psi_1,\Psi_2,M_k,p_{kv}).$$

A cubature formula with nodes M_k^* and weights p_{kv}^* is called optimal, asymptotically optimal, optimal with respect to order on the class of functions Ψ_1 and surfaces Ψ_2, if

$$\frac{R_n(\Psi_1,\Psi_2,M_k^*,p_{kv}^*)}{\zeta_n[\Psi_1,\Psi_2]} = 1, \sim 1, \asymp 1,$$

respectively.

Let $\Psi_1 = H_\alpha(1),\ 0 < \alpha \leqslant 1$, and $\Psi_2 = L_1(B,\beta)\ 0 < \beta \leqslant 1$. Let us construct an optimal with respect to order method for calculating integrals (7.1) on the classes of functions Ψ_1 and surfaces Ψ_2. Let $S(G)$ be a "square" of the surface G. Divide the surface G into n parts $g_k,\ k = 1, 2, \ldots, n$, so that a "square" of each of the domains g_k has the area of order $|S(G)|/n$, where $|S(G)|$ is the area of $S(G)$. We take a point M_k in each of domains g_k at the center of the domain g_k.

Calculate integral (7.1) by the formula

$$Jf = \sum_{k=1}^{n} f(M_k) \iint\limits_{g_k} \frac{dS}{\Big((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\tau_3 - t_3)^2\Big)^\lambda} + R_n(f, G). \tag{7.3}$$

Theorem 7.1. *Formula (7.3), has the error*

$$R_n(\Psi_1, \Psi_2) \asymp n^{-\alpha/2},$$

and is optimal with respect to order on the classes $\Psi_1 = H_\alpha,\ 0 < \alpha \leqslant 1$, *and* $\Psi_2 = L_1(B,\beta),\ 0 < \beta \leqslant 1$, *among all formulas (7.2) with* $\rho = 0$.

Proof. Assume for simplicity that the surface G is given by the equation $z = \varphi(x, y),\ (x, y) \in G_0,\ \varphi(x, y) \geqslant 0$. Let $\varphi_x(x, y) := p,\ \varphi_y(x, y) := q$. Write the integral Jf as

$$Jf = \iint\limits_{G_0} \frac{f(\tau_1, \tau_2, \varphi(\tau_1, \tau_2))\sqrt{1 + p^2(\tau_1, \tau_2) + q^2(\tau_1, \tau_2)}d\tau_1 d\tau_2}{\Big[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2\Big]^\lambda}. \tag{7.4}$$

The function $f(\tau_1, \tau_2, \varphi(\tau_1, \tau_2))$ belongs to the Hölder class H_α over G_0, and the function $\frac{\sqrt{1+p^2+q^2}}{\Big[(\tau_1-t_1)^2+(\tau_2-t_2)^2+(\varphi(\tau_1,\tau_2)-\varphi(t_1,t_2))^2\Big]^\lambda}$ is positive.

Let $M_k = (m_1^k, m_2^k, m_3^k)$ be the nodes of cubature formula (7.2). Let $\psi(\tau) := (d(\tau, \{M_k\}))^\alpha$, where $d(\tau, \{M_k\})$ is the distance between the point τ and the set of the nodes $\{M_k\}$, where the distance is measured along the geodesics of the surface G. This distance satisfies the Hölder condition $H_\alpha(1)$. Hence the function $\psi^*(\tau_1, \tau_2) = \psi(\tau_1, \tau_2, \varphi(\tau_1, \tau_2))$ belongs to the Hölder class $H_\alpha(A)$ and vanishes at the nodes $(m_1^k, m_2^k),\ k = 1, 2, \ldots, n$. Thus,

$$\begin{aligned}
\zeta_n(\Psi_1, \Psi_2) &\geqslant \frac{1}{S(G_0)} \iint\limits_{G_0} \iint\limits_{G_0} \frac{\psi(\tau_1, \tau_2, \varphi(\tau_1, \tau_2))\sqrt{1 + p^2 + q^2}d\tau_1 d\tau_2 dt_1 dt_2}{\Big[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2\Big]^\lambda} \geqslant \\
&\geqslant \frac{1}{S(G_0)} \iint\limits_{G_0} \psi(\tau_1, \tau_2, \phi(\tau_1, \tau_2)) d\tau_1 d\tau_2 \times \\
&\times \min_t \iint\limits_{G_0} \frac{\sqrt{1 + p^2 + q^2}}{\Big[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2 + (\varphi(\tau_1, \tau_2) - \varphi(t_1, t_2))^2\Big]^\lambda} d\tau_1 d\tau_2 \geqslant \\
&\geqslant \frac{A}{n^{\alpha/2}} \min_t \iint\limits_{G} \frac{ds}{\big(r(t, \tau)\big)^\lambda},
\end{aligned}$$

where $S(G_0)$ is the "square" of the surface G_0.

Therefore the error of formula (7.3) is estimated by the inequality $R_n \leqslant \frac{A}{n^{\alpha/2}}$.

Theorem 7.1 is proved. ■

Remark 1. *The method of decomposition of the domain G into smaller parts $g_k,\ k = 1, 2, \ldots, n$, described below, is optimal with respect to order for classes of functions $\Psi_1 = H_\alpha,\ 0 < \alpha \leqslant 1$, and of surfaces $\Psi_2 = L_0(B, \beta),\ 0 < \beta \leqslant 1$ for $\alpha \leqslant \beta$.*

Remark 2. *From formula (7.4) it follows that if the function $f \in W^{r,r}(1)$ and the surface $G \in L_s(B, \alpha)$, then the function $f(\tau_1, \tau_2, \varphi(\tau_1, \tau_2)) \in W^{v,v}(A)$, where $v = \min(r, s)$. Therefore, repeating the above arguments, one proves that the accuracy of calculation of integral (7.4) by cubature formulas using n values of integrand function does not exceed $O(n^{-v/2})$.*

From this remark it follows that if the surface G consists of several parts, for example of surfaces G_1 and G_2 having common edge L, then it is necessary to calculate the integrals for the surface G_1 and the surface G_2 separately. If the surface G is divided into smaller parts $g_k,\ k = 1, 2, \ldots, n$, the domains g_k, the curve L passes inside of these domains, should be associated with the class of surfaces $L_0(B, 1)$. In these domains the accuracy of calculation of the integral does not exceed than $O(n_k^{-1})$, where n_k is the number of nodes of the cubature formula used in the domain g_k.

For this reason the cusps and the nodes, in which three or more domains G_k, which are parts of the domain G touch each other, must belong to the boundaries of the covering domains $g_k,\ k = 1, 2, \ldots, n$.

The universal code for computing the capacitances, described in Section 9, is based on optimal with respect to order cubature formulas for calculating integrals on the classes of functions $H_\alpha,\ 0 < \alpha \leqslant 1$ and of surfaces $L_0(B, \beta),\ \ B = const,\ \alpha \leqslant \beta,\ \beta \leqslant 1$.

The algorithm constructed in Section 9 is optimal on this class of surfaces and does not require special treatment of edges and conical points of the surface.

When one studies cubature formulas on the classes $W^{r,r}(A)$, $r > 1$, and $L_s(B, \beta)$, $s \geqslant 1,\ 0 \leqslant \beta \leqslant 1$, one has to develop a method to compute accurately the integrals in a neighborhood of the above singular points of the surface.

8. Calculation of weights of cubature formulas.

In calculating weakly singular integrals by cubature formulas (6.8) it is necessary to calculate integrals of the form of

$$J_{kl}(t_1, t_2) = \int\limits_{\Delta_{kl}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda}$$

for different values $(t_1, t_2) \in [-1, 1]^2$.

Let $(t_1, t_2) \in \Delta_{ij}$. Let us consider two possibilities: 1) the square Δ_{kl} and the square Δ_{ij} have nonempty intersection; 2) the square Δ_{kl} is does not have common points with the square Δ_{ij}.

First consider the second case, when the function

$$\varphi(\tau_1, \tau_2) = \frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda},$$

is smooth. Here $(\tau_1, \tau_2) \in \Delta_{kl}$, and $(t_1, t_2) \in \Delta_{ij}$.

In this case one has:

$$\left|\frac{\partial^r \varphi(\tau_1, \tau_2)}{\partial \tau_1^r}\right| \leq \frac{r! 2^{2r}}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda + r/2}}$$

and, if the squares Δ_{kl} and Δ_{ij} do not have common points, one gets:

$$\left|\frac{\partial^r \varphi(\tau_1, \tau_2)}{\partial \tau_1^r}\right| \leq \frac{2^r r! n^{2\lambda + r}}{2^\lambda}.$$

Similar estimates holds for partial derivative with respect to τ_2.

Calculate the integral $J_{kl}(t_1, t_2)$ by the Gauss cubature formula:

$$J_{kl}(t_1, t_2) = \int\limits_{\Delta_{kl}} P_{mm} \left[\frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda}\right] d\tau_1 d\tau_2 + R_{mm}(\Delta_{kl}),$$

where $P_{mm} = P_m^{\tau_1} P_m^{\tau_2}$, $P_m^{\tau_i}$ $(i = 1, 2)$ is the projection operator onto the set of interpolation polynomials of degree m with nodes at the zeros of the Legendre polynomial, which maps the segment $[-1, 1]$ onto the segment $[x_k, x_{k+1}]$ for $i = 1$, and onto the segment $[x_l, x_{l+1}]$ for $i = 2$.

An integer m is chosen so that $|R_{mm}| \leq n^{-2-\alpha}$ for cubature formulas on the Hölder class $H_{\alpha\alpha}$, and $|R_{mm}| \leq n^{-r-\alpha}$ for cubature formulas on the class W^{rr}.

This requirement is made because the error of calculation of the coefficients $J_{kl}(t_1, t_2)$ must not exceed the error of formula (6.5).

Using r derivatives of the integrand in the error $R_{mm}(\Delta_{kl})$, one gets:

$$|R_{mm}(\Delta_{kl})| \leq \frac{B_r 2^r r!}{m^{r-1}} \left(\frac{2}{n}\right)^{2-2\lambda},$$

where B_r is the constant appearing in Jackson's theorems. It is known that the constants B_r are bounded by a constant, denoted b, uniformly with respect to r. In the case of periodic functions $b = 1$ ([18]), and in the general case b is apparently unknown.

If $r = 2$ and $m = B_r 2^r r! n^{2\lambda}$, then one gets the error estimate given for cubature formula (6.5).

Now, consider a method for calculating the integrals $J_{kl}(t_1, t_2)$ when the square Δ_{kl} has nonempty intersection with the square Δ_{ij}. For definiteness we consider the calculation of the integral $J_{ij}(t_1, t_2)$ by the formula:

$$J_{ij}(t_1, t_2) = \int\limits_{\Delta_{ij}} P_{mm} \left[\frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^\lambda + h}\right] d\tau_1 d\tau_2 + R_{mm}(\Delta_{ij}),$$

where $h = const > 0$ will be specified below.

One has:

$$|R_{mm}(\Delta_{ij})| \leq h \int\limits_{\Delta_{ij}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda} \left(((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda} + h \right)} +$$

$$+ \int\limits_{\Delta_{ij}} D_{mm} \left[\frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda} + h} \right] d\tau_1 d\tau_2 = r_1 + r_2,$$

where $D_{mm} = I - P_{mm}$, and I is an identity operator, and

$$r_1 \leq h \int\limits_{\Delta_{ij}} \frac{d\tau_1 d\tau_2}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{(\lambda+1)/2} \left(((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda} + h \right)^{(1+\lambda)/2}} \leq$$

$$\leq \frac{2\pi}{1-\lambda} h^{(1-\lambda)/2} \left(\frac{2\pi}{n} \right)^{1-2\lambda}. \tag{8.1}$$

The function $\frac{1}{((\tau_1 - t_1)^2 + (\tau_2 - t_2)^2)^{\lambda} + h}$ is infinitely smooth. Using bounds for its first derivatives for $\lambda \geq 1/2$, one gets:

$$r_2 \leq \frac{8\lambda B_1}{n^4 h^2 m}. \tag{8.2}$$

From inequality (8.1) it follows that for getting accuracy $O(n^{-1-\alpha})$ one has to have $h = n^{-2(2\lambda+\alpha)/(1-\lambda)}$ and from inequality (8.2) it follows that one has to have $m = max([n^{(8\lambda+4\alpha)/(1-\lambda)+\alpha-3}], 1)$.

9. Iterative methods for calculating electrical capacitancies of conductors of arbitrary shapes.

Numerical methods for solving electrostatic problems, in particular, calculating capacitancies of conductors of arbitrary shapes, are of practical interest in many applications. Electrostatic problems solvable in closed form are collected in [19,20,21]. Some of the problems were solved in closed form using integral equations, Wiener-Hopf and singular integral equations [22]. Electrostatic problems for a finite circular hollow cylinder (tube) were studied in [23] by numerical methods. In [24] the variational methods of Ritz and Trefftz are discussed. Galerkin's and other projection methods are studied in [25]. In practice these methods are time-consuming and variational methods in three-dimensional static problems probably have some advantages over the grid method. There exists a vast literature on calculation of the capacitances of perfect conductors [20,26]. In [20] there is a reference section which gives the capacitance of the conductors of certain shapes (more than 800 shapes are considered in [20]). In [26] and [27] a systematic exposition of variational methods for estimation of the capacitances is given. In [28] there are some programs for calculating the two-dimensional static fields using integral equations method. In monograph [7] iterative methods for solving interior and exterior boundary value problems in electrostatics are proposed and mathematically justified. Upper and lower estimates for some functionals of electrostatic fields are obtained in [7]. Such functionals are the capacitances of perfect conductors and the polarizability tensors of bodies of arbitrary shape.

These bodies are described by their dielectric permittivity, magnetic permeability and conductivity. They can be homogeneous or flaky. The main point is: these bodies have arbitrary geometrical shapes.

The methods, developed in [7], yield analytical formulas for calculation of the capacitances and polarizability tensors of bodies of arbitrary shapes with any given accuracy. Error estimates for these formulas are obtained in [7]. We give here the formulas for calculating the capacitances of the conductors of arbitrary shapes [7]:

$$C^{(n)} = 4\pi\varepsilon_0 S^2 \left\{ \frac{(-1)^n}{(2\pi)^n} \int_\Gamma \int_\Gamma \frac{dsdt}{r_{st}} \underbrace{\int_\Gamma \dots \int_\Gamma}_{n \text{ times}} \psi(t,t_1)\dots\psi(t_{n-1},t_n)dt_1\dots dt_n \right\}^{-1},$$

where S is the surface area of the surface Γ of the conductor, ε_0 is the dielectric constant of the medium, $r_{st} := |s-t|$, and $\psi(t,s) := \frac{\partial}{\partial N_t}\frac{1}{r_{st}}$,

$$C^{(0)} = \frac{4\pi\varepsilon_0 S^2}{J} \le C, \quad J \equiv \int_\Gamma \int_\Gamma \frac{dsdt}{r_{st}}, \quad S = \text{meas}\Gamma.$$

It is proved in [7] that

$$\left| C - C^{(n)} \right| \le Aq^n, \quad 0 < q < 1,$$

where A and q are constants which depend only on the geometry of Γ.

We use these formulas are used to construct the computer code for calculating the capacitances of the conductors of arbitrary shapes.

It is proved in [7], that

$$C^{(n)} = 4\pi\varepsilon_0 S^2 \left(\int\limits_\Gamma \int\limits_\Gamma r_{st}^{-1}\delta_n(t)dtds \right)^{-1}, \tag{9.1}$$

where δ_n is defined by the iterative process:

$$\delta_{n+1} = -A\delta_n, \ \delta_0 = 1, \int\limits_\Gamma \delta_n dt = S, \tag{9.2}$$

and A is defined by the formula:

$$A\delta = \int\limits_\Gamma \delta(t) \frac{\partial}{\partial N_s} \frac{1}{2\pi r_{st}} dt,$$

where N_s is the outer unit normal to Γ at the point s.

To use iterative process (9.2), one has to calculate the weakly singular integral

$$\frac{1}{2\pi} \int\limits_\Gamma \delta(t) \frac{\partial}{\partial N_S} \frac{1}{r_{st}} dt. \tag{9.3}$$

Let us describe the construction of a cubature formula for calculating integral (9.3), assuming for simplicity that the domain G, bounded by the surface Γ, is convex. This asumption can be removed.

Let $\mathbb{S}$ be the inscribed in the conductor sphere of maximal radius r^*, centered at the origin. Introduce the spherical coordinates system (r, ϕ, θ), and the set of the nodes (r^*, ϕ_k, θ_l), where $\phi_k = 2k\pi/n,\ k = 0, 1, \dots, n,\ \theta_l = \pi l/m,\ l = 0, 1, \dots, m$. Assume that m is even, and cover the sphere $\mathbb{S}$ with the spherical triangles $\Delta_k,\ k = 1, 2, \dots, N,\ N = 2n(m-1)$.

Let us describe the construction of the spherical triangles. For $0 \le \Theta \le \pi/m$ the triangles $\Delta_k, k = 1, 2, \dots, n$ have vertices $(r^*, 0, 0)$, $(r^*, \phi_{k-1}, \theta_1)$, (r^*, ϕ_k, θ_1), $k = 1, 2, \dots, n$.

For $\theta_l \le \theta \le \theta_{l+1},\ l = 1, 2, \dots, m/2 - 1$, the triangles $\Delta_k, k = n + 2n(l-1) + j,\ 1 \le j \le 2n$ are constructed as follows. The rectangle $[0, 2\pi; \theta_l, \theta_{l+1}]$ is covered with the squares $\Delta_{kl} = [\phi_k, \phi_{k+1}; \theta_l, \theta_{l+1}],\ k = 0, 1, \dots, n-1$. Each of the squares Δ_{kl} is divived into two equal triangles Δ^1_{kl} and $\Delta^2_{kl},\ k = 0, 1, \dots, n-1,\ l = 1, 2, \dots, m/2 - 1$. The spherical triangles Δ^1_{kl} and $\Delta^2_{kl},\ k = 0, 1, \dots, n-1,\ l = 1, 2, \dots, m/2 - 1$, are images of triangles Δ^1_{kl} and Δ^2_{kl} on the sphere $\mathbb{S}$

As a result of these constructions the sphere $\mathbb{S}$ is covered with triangles $\Delta_k,\ k = 1, 2, \dots, N$.

We draw the straight lines through the origin and vertices of the triangle $\Delta_k,\ k = 1, 2, \dots, N$. The points of intersection of these lines with the surface Γ are vertices of the triangle $\overline{\Delta}_k,\ k = 1, 2, \dots, N$. As a result of these constructions the surface Γ is approximated by the surface Γ_N consisting of triangle $\overline{\Delta}_k,\ k = 1, 2, \dots, N$, and integral (9.3) is approximated by the integral

$$U(s) = \int\limits_{\Gamma_N} \delta(t) \frac{\partial}{\partial N_S} \frac{1}{r_{st}} dt. \tag{9.4}$$

We fix each triangle $\overline{\Delta}_k,\ k = 1, 2, \dots, N$, and associate with it a point $\tau_k \in \overline{\Delta}_k,\ k = 1, 2, \dots, N$, equidistant from the vertices of the triangle $\overline{\Delta}_k,\ k = 1, 2, \dots, N$. We calculate integral (9.4) at the points $\tau_k,\ k = 1, 2, \dots, N$, by the cubature formulas constructed in paragraphs 5-7 for the Hölder classes. After calculating the values $U(\tau_k),\ k = 1, 2, \dots, N$ by these cubature the integral

$$\tilde{C}^{(1)} = -4\pi\varepsilon_0 S_N^2 \left(\int\limits_{\Gamma_N} \int\limits_{\Gamma_N} r_{st}^{-1} \tilde{U}(t) dt ds \right)^{-1}$$

is calculated, where $\tilde{U}(t) = U(\tau_k)$ for $t \in \overline{\Delta}_k,\ k = 1, 2, \dots, N$, S_N is area of the surface Γ_N, $\tilde{C}^{(1)}$ is approximation to the value of $C^{(1)}$. The successive iterations are calculated analogously.

10. Numerical examples.

In this section the numerical results are given. As an example we calculated the capacitances of various ellipsoids, because for ellipsoids one knows ([19]) the analytical formula for the capacitance, which makes it possible to evaluate the accuracy of the numerical results. Consider the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

It is known [19,20] that the exact value of the capacitance of ellipsoid with $a = b$ is:

$$C = \frac{4\pi\varepsilon_0 \sqrt{(a^2 - c^2)}}{\arccos(c/a)}.$$

Let $a = b = 1$, and $\varepsilon_0 = 1$. We have calculate the capacitance C for different values of the semiaxis c. The results of the calculations are given in Table 1.

It is known ([7], p.43), that the capacitance of a metallic disc of radius a is $C = 8a\varepsilon_0$, and one can see from Table 1, that asymptotically, as $c \to 0$, this formula can be used practically for the ellipsoids with $c \leq 0.001$ with the error about 0.005.

Table 1

C	n	m	N	Exact value	Error	Relative error	Calculation time
0.9	40	30	2320	12.144630	-0.221200	0.018212	25 sec
0.5	40	30	2320	10.392304	-0.222042	0.021366	25 sec
0.1	40	30	2320	8.5020638	-0.301189	0.035425	25 sec
0.01	40	30	2320	8.050854	0.072132	0.008959	25 sec
0.001	40	30	2320	8.005092	-0.821528	0.106374	25 sec
0.0001	40	30	2320	8.000509	-1.068178	0.133513	25 sec
0.9	50	40	3900	12.144630	-0.180510	0.014801	1 min 15 sec
0.5	50	40	3900	10.392304	-0.185642	0.017860	1 min 15 sec
0.1	50	40	3900	8.5020638	-0.288628	0.033947	1 min 15 sec
0.01	50	40	3900	8.050854	-0.372047	0.046212	1 min 15 sec
0.001	50	40	3900	8.005092	-0.586733	0.073295	1 min 15 sec
0.0001	50	40	3900	8.000509	-0.933288	0.116653	1 min 15 sec
0.9	60	50	5880	12.144630	-0.152009	0.012516	4 min
0.5	60	50	5880	10.392304	-0.160023	0.015391	4 min
0.1	60	50	5880	8.5020638	-0.283364	0.033328	4 min
0.01	60	50	5880	8.050854	0.532250	0.061110	4 min
0.001	60	50	5880	8.005092	-0.391755	0.048939	4 min
0.0001	60	50	5880	8.000509	-0.880394	0.110042	4 min

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